

Gromov-Witten Theory and Threshold Corrections

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July 2004

Abstract

We present an overview of Gromov-Witten theory and its links with string theory compactifications, focussing on the GW potential as the generating function for topological string amplitudes at genus g . Restricting to Calabi-Yau target spaces, we give a complete derivation of the GW potential, discuss problems of multicovers and the infinite product expression. We explain the link with counting instantons or BPS states in type IIA and heterotic string theories. We show why the numbers of BPS states on the heterotic side can be a priori expressed in terms of those on the type IIA side, and vice versa. We compute heterotic one-loop integrals to obtain the genus g GW potential, and detail two ways to obtain threshold corrections for heterotic orbifolds, a prerequisite for the notorious work by Harvey and Moore. We review this long and cumbersome construction in a self-contained way and make it explicit in examples of compactifications. We also develop the relation to Jacobi forms and automorphic forms, and clarify the meaning of the Gopakumar-Vafa invariants.

arXiv:hep-th/0605087 v1 8 May 2006

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Introduction

Gromov-Witten (GW) theory is usually considered to have its origin in Witten's seminal work [W-91] on $2d$ gravity where he solved integrals of algebraic geometry having the enumerative meanings of counting instantons (or holomorphic curves embedded in a target space). The thirteen-year-old theory has not merely reached the age of reason, it has also escalated into an independent and explosive field of study. What was originally a string theoretic problem and approach, has now almost been hijacked by algebraic geometers, subdued to rigour and held to leading strings. And gladly. For the theory now stands on a firm rock, albeit an arduous one to clamber for the. This makes it difficult for physicists nowadays to understand what was once a topic of their own, of instanton sums in topological field theory.

GW Theory

In GW theory, one studies holomorphic maps $f : \Sigma_g \rightarrow X$ from a Riemann surface Σ_g to a target space X , also known as instantons. These give non-perturbative contributions to the action of an $N = 2$ sigma-model defined on X whose twisting yields a topological field theory (A-model). The latter's amplitudes $F_g(t)$ at genus g are called topological string partition functions in the sense that they do not depend on the worldsheet (Σ_g) metric but only on the cohomology class of the spacetime metric (Kähler moduli t of the target).

The image of the map will be a holomorphic curve with a homology class, singular points, an arithmetic genus, a geometric genus, etc. The aim is to “count” such objects in X ; it turns out it is easier to “count maps” with domain a well-behaved Riemann surface (that possibly degenerates). Roughly speaking, the “number of maps” corresponds to the GW invariant (a rational number), while the “number of (image) curves” is our instanton number (an integer).

Of course, these maps or curves come typically in infinite families, hence the need to impose constraints such as: the curve should pass through a given hypersurface Z of X , and its cohomological dual $Z^* = \gamma$ is a 2-form. An appropriate array of constraints thus yields hopefully a finite number, though a rational one, as the moduli spaces we work with are orbifolds and there will be a trailing denominator from the order of some automorphism group. From these rational numbers one can extract *integer numbers* by a recursion involving *excess intersection*; these are our sought-for instanton numbers.

For *closed* string instantons, and we shall only consider such, the physical approach for counting them is paralleled by a well-defined theory developed by algebraic geometers. The latter use moduli spaces of stable maps, on which they integrate pull-backs of differential forms γ_i on the target space, to define the *GW invariants*. *Calabi-Yau threefolds* are singled out in nature as not requiring any constraints (to obtain a finite number of maps) and hence requiring no γ_i either. Their GW invariants thus only depend on the genus g of the domain curve and the homology class β (or d) of the image curve; we denote them by N_d^g or $\langle 1 \rangle_{g,\beta}$.

These rational numbers are linearly related to the *instanton numbers* (or *BPS invariants*) n_d^g . That the latter are *integers* is a non-trivial fact met in all examples known so far, shadowy for mathematicians but natural for physicists who see in them numbers of D-branes (or BPS states) wrapped around particular cycles of the CY threefold.

GW Potential for Type IIA and Heterotic Strings

The GW invariants at genus g can be gathered into *GW potentials* F_g . This is nothing but the genus g amplitude of the afore-mentioned topological strings. The closest string theory among the five $10d$ theories to describe this model is *type IIA*, and in this context the F_g play the role of *couplings* to the graviphoton field strength in the action. They are exact at genus g , and can

thus be computed at strong coupling, also known as the decompactification limit or “M-theory” (11-dimensional).

In the context of M-theory, a one-loop Schwinger integral will yield the exact result for the F_g (though g -loop was required in type IIA), much as Seiberg-Witten theory is exact at one-loop. Because of strong coupling, only the lightest states will contribute to the integral: D0- and D2 branes of type IIA, corresponding to KK-modes and M2-branes of M-theory. The *full* GW potential for the example of the local conifold can then be re-written in product-notation and yields the generating function for plane partitions. Generalisations of this example, including spin, can equally be presented in the form of infinite products and it is hoped that these have *automorphic properties* of some sort. The powers occurring in these products contain our sought-for instanton numbers and go here by the name of Gopakumar-Vafa invariants.

Another context where the F_g are exact at one-loop is *heterotic string theory*, which is related to type IIA by *duality*. The topological amplitudes also have the meaning of couplings to the graviphoton field strength in the action, yet the one-loop integral can be computed via the trick of lattice reduction, yielding awkward expressions but whose holomorphic limit recovers the desired constant term of the instanton generating function (this term arising from constant maps is related to the *global* topology of the space and appears thus with the Euler character).

Other prowesses from calculations in heterotic theory involve the full determination of the prepotential, or F_0 . It goes via an ODE for the gauge coupling that contains integrals of a theta function and a modular form over the fundamental domain $\mathcal{F} = \mathcal{H}/SL(2, \mathbb{Z})$. The latter integrals are resolved by the trick of unfolding the fundamental domain and yield an astounding expression: the logarithm of automorphic products à-la Borcherds. Reading off this result as if we were in type IIA string theory, we can extract our instanton numbers or GW invariants. In other words, there is a salient relationship between GW invariants and automorphic forms – at least for the genus-0 invariants counting *rational* curves in the target space. Whether this holds for higher-genus invariants remains opaque.

Yet the string duality relating type IIA and heterotic theories still provides an original way for presenting enumerative topological information as powers in infinite product expressions: on the type IIA side we have linear combinations of Gopakumar-Vafa invariants, while on the heterotic we have powers of Borcherds products (that originate from coefficients of nearly-holomorphic modular forms) and that are parametrised by “positive” roots of even self-dual Lorentzian lattices.

Bella Vista around the GW Potential

The structures and mathematical tools hidden behind this long chain of dualities, weak- and strong coupling limits and SCFTs are quite enticing. For instance, Borcherds’ construction of automorphic forms goes via lifting of *Jacobi forms* of index 1 (which are roughly isomorphic to modular forms), and if the latter have zero weight the former admit a product expression. Examples of Jacobi forms include the *elliptic genus*, a physical-topological object defined by a trace over the Hilbert space of states of an SCFT: $\Phi(q, y) = \text{tr}_{R,R}(-1)^F q^{L_0 - \frac{c}{24}} y^{J_0}$. It has an explicit expression known for several compactification spaces (CY 1,2,3,4-folds).

For a non-linear sigma model on the complex surface $K3$ (CY two-fold), the above traces (which can be generalised for all combinations of R or NS in the left- and right-moving sectors) yield several topological indices (elliptic genus, Dirac genera, partition functions, N=4 characters) that can be written as sums over a finite number of ‘orbits’ of the N=4 SCA.

The starting point for the one-loop integrals in the heterotic context are *threshold corrections* to gauge couplings. These can be computed in the effective field theory and the integrand will be essentially a trace over the internal degrees of freedom. At this stage there is a road junction for the special example of heterotic compactification on $K3 \times T^2$: we can either directly compute the

partition function of the model and find the trace over the internal theory to yield $\Gamma_{10,2}E_6/\eta^{24}$, or we can rewrite the abstract integrand using heavy algebra and landing on the so-called “new susy index” (a variant of the elliptic genus, closely related to *helicity supertraces*). From here, we can anew compute the explicit value of the integrand using techniques from helicity supertraces over BPS states (which tell us in particular that vector- and hypermultiplets contribute equal and opposite amounts to the supertrace), and we arrive at the same result of $\Gamma_{10,2}E_6/\eta^{24}$. This is yet without the additional factor from the gauge group, a term involving quadratic Casimir operators and beta functions.

In other words, the path towards writing down a one-loop threshold correction, solving it and presenting it in a shape suitable to obtain Gromov-Witten invariants is long and cumbersome, requires patience from the reader as well as familiarity with many topics of string theory and geometry. It is the fondest hope of the author that this long path is brought forward in the present article in a self-reliant and comprehensible manner.

Acknowledgements

This paper was written two years ago; I have not found time to complete it and never will. It is a pleasure to remember the many people who have helped me with their valuable comments. I acknowledge a myriad of discussions that have contributed to my understanding of the subject and of related topics during the past years, and shall only mention a few names: J. Bryan, R. Pandharipande, M. Roček and S. Katz from the PCMI school on Mirror Symmetry; J. de Boer, A. Klemm and B. Pioline on different occasions; and especially R. Dijkgraaf.

I thank prof. Lüst for his hospitality at the Humboldt University in Berlin (support from a ‘Kurzfristiges Forschungsstipendium’ of the DAAD), as well as prof. Sossinsky for his hospitality at the Independent University of Moscow (support from the Netherlands Organisation for Scientific Research (NWO)), and most of all the Racah Institute of the Hebrew University whose financial support enabled me to work on this project within Israel’s vibrant string theory community.

1 Gromov-Witten Results on CY Threefolds

This section restricts to Calabi-Yau threefolds (Kähler manifolds with vanishing first Chern class). The first two subsections deal with the general theory (also valid for non-CY varieties) of GW invariants; we recommend [P-00] and [CK-99] for further details. From the third subsection on, the formulae only make sense for CY threefolds: we build the potentials F_g and $F = \sum_g F_g \lambda^{2g-2}$ based on the previous abstract definition of GW invariants, but in fact most of it can be understood independently. Though initially rather mathematical, this section later introduces physical concepts such as the potential, the Yukawa couplings, etc, all relating to the framework of *closed* string theory.

1.1 The moduli space of stable maps

Gromov-Witten theory studies the properties of curves embedded in larger spaces via a holomorphic map $f : \Sigma_{g,n} \rightarrow X$. We start by introducing Kontsevich’s powerful moduli space of stable maps, an extension of the idea of moduli space $\overline{\mathcal{M}}_{g,n}$ of stable nodal curves $\Sigma_{g,n}$ of genus g with n markings p_1, \dots, p_n . Recall that $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$ ($g > 1$).

A **stable map** $f : \Sigma_{g,n} \rightarrow X$ is a map from a pointed nodal curve to a variety X such that the contracted components (*i.e.* where f is constant) of genus 0 have 3 special points and those of genus 1 have 1 special point. (This ensures there won’t be any infinitesimal automorphisms, ie the automorphism group should be at most finite.)

The **moduli space** of stable maps is

$$\overline{\mathcal{M}}_{g,n}(X, \beta) := \{ \text{stable maps } f : \Sigma \longrightarrow X \mid f_*([\Sigma]) = \beta \in H_2(X, \mathbb{Z}) \} / \sim.$$

Its elements are denoted $(\Sigma, p_1, \dots, p_n, f)$, and this is *isomorphic* to $(\Sigma', p'_1, \dots, p'_n, f')$ iff $\exists \tau : \Sigma \rightarrow \Sigma'$ with $\tau(p_i) = p'_i$ and $f' \circ \tau = f$.

Note that $\beta = 0$ means constant maps. If $H_2(X, \mathbb{Z}) = \mathbb{Z}$ (this is the case of a hypersurface X in \mathbb{P}^N – by the Lefschetz hyperplane theorem), then we shall rather use the integer d to label the class $\beta = d \cdot \ell$ (where ℓ is the generator of H^2). Note also that a stable map need not have a stable domain curve, except if $\beta = 0$ where stable maps are equivalent to stable curves.

Example 1.1.

$$\begin{aligned} \overline{\mathcal{M}}_{g,n}(X, 0) &= \overline{\mathcal{M}}_{g,n} \times X. \\ \overline{\mathcal{M}}_{g,n}(\text{point}, 0) &= \overline{\mathcal{M}}_{g,n} \\ \overline{\mathcal{M}}_{0,0}(X, 0) &= \overline{\mathcal{M}}_{0,0} \times X = \emptyset \times X = \emptyset \\ \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 1) &= \text{point} \end{aligned}$$

Note that $\mathcal{M}_{g,n}(X, \beta)$ includes degree d coverings of the line (always singular for $d > 1$, as the RH formula implies branch points).

Example 1.2. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^N, 1) = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^N, 1) = G(\mathbb{P}^1, \mathbb{P}^N)$, the Grassmannian of lines in \mathbb{P}^N , of dimension $2N - 2$. Note that $g = 0$ implies that the geometric genus of $\Sigma_{0,0}$ also vanishes, leaving only a tree of \mathbb{P}^1 's as a possible candidate (if the \mathbb{P}^1 's form a loop, the arithmetic genus g will increase by 1). Together with $n = 0$ and $d = 1$, this forbids contractions (which would be unstable). Embedding a tree in \mathbb{P}^N means the total degree is the sum of the degrees of each component, *i.e.* the number of lines in the tree; thus we can have only 1 line.

Example 1.3. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d)$, $d > 1$, is the space of all d -fold covers of the line, and has dimension $2(d-1)$ (all branch points). This agrees with the dimension of the space of parametrised rational curves¹ of degree d to \mathbb{P}^1 . The boundary of this moduli space consists of all maps from trees with at least two lines (hence a nodal curve); adding contractions, the domain tree can have more than d lines and the configuration can be very wild.

We now define the *evaluation map*

$$\begin{aligned} \text{ev}_i : \quad \overline{\mathcal{M}}_{g,n}(X, \beta) &\rightarrow X \\ (\Sigma, p_1, \dots, p_n, f) &\mapsto f(p_i) \end{aligned}$$

which allows us to present the *universal curve*

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{g,n+1}(X, \beta) & \\ & \pi \swarrow \quad \searrow \text{ev}_{n+1} & \\ \overline{\mathcal{M}}_{g,n}(X, \beta) & & f(p_{n+1}) \in X \end{array} \quad (1.4)$$

¹In general, a (parametrised) *rational curve* of degree d in \mathbb{P}^N is a tuple of homogeneous polynomials of degree d in x_0, x_1 : $(f_0(x_0, x_1), \dots, f_N(x_0, x_1))$, where the f_i have no common factors. The curve will be degenerate if it lies in a hyperplane; this happens in particular when some f_i is 0 (hyperplane is $\{y_i = 0\}$), or when $f_i = f_j$ (hyperplane is $\{y_i - y_j = 0\}$), or in more general situations like $(x_0^2, x_1^2, -x_0^2 - x_1^2)$ (hyperplane is the line $\{y_0 + y_1 + y_2 = 0\}$ in the plane, and the map is a double cover branched over $(1, 0, -1)$ and $(0, 1, -1)$), etc.

Rational curves can be nodal (increasing their arithmetic genus), as is the case of $(x_0^3, x_0x_1^2, x_1(x_1+x_0)(x_1-x_0))$, also written $(1, z^2, z(z+1)(z-1)) = (1, x, y)$ or $y^2 = x(x-1)^2$, which is the cubic with an ordinary double node in $(1, 1, 0)$, *i.e.* an elliptic curve ($g = 1$).

The dimension of this space of rational curves is $\frac{(N+1)(d+1)-4}{2}$ (number of coefficients in the polynomials), where we have subtracted 1 for the redundancy in \mathbb{P}^N and 3 for the reparametrisations of (x_0, x_1) . This number is much smaller than the dimensionality of all degree d curves in \mathbb{P}^N , which are parametrised by $\binom{N+d}{d} - 1$ coefficients (corresponding to polynomials of degree d in $N+1$ variables). The two dimensions match for $N = 2$ and $d = 1, 2$, *i.e.* all lines and conics in the plane can be parametrised as rational curves; for plane cubics, quartics, ..., only a minority of them are rational. Conversely, a rational curve of degree d in \mathbb{P}^2 will in general be a plane curve of degree $D \geq 2d - 3$ (need to solve $dD + 1$ equations for $d(d+3)/2$ projective unknowns).

with which we will pull back objects from X to $\overline{\mathcal{M}}_{g,n}(X, \beta)$ (a standard trick).

The space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a stack (orbifold), usually singular and almost always will its deformation theory be obstructed. It can consist of several components of different dimension, so we can only speak of an expected or *virtual dimension*, vdim , which is a lower bound to the actual local dimension. If all dimensions agree, the theory is *unobstructed* and $\text{vdim} = \dim$. This number vdim is defined as the deformations $h^0(\Sigma, f^*N^{\Sigma/X})$ of the image curve $f(\Sigma)$ and is easily computed via Riemann-Roch in case of no obstruction [KM-94]:

$$\text{vdim } \overline{\mathcal{M}}_{g,n}(X, \beta) = \int_{\beta} c_1(X) + (\dim X - 3)(1 - g) + n. \quad (1.5)$$

Example 1.6. When $\beta = 0$, $\text{vdim} = \dim X(1 - g) + \dim \overline{\mathcal{M}}_{g,n}$, which matches the actual dimension ($\dim X + \dim \overline{\mathcal{M}}_{g,n}$) only for $g = 0$. Thus constant maps are obstructed at $g > 0$.

Example 1.7. If X is a CY threefold (i.e. $c_1(X) = 0$), the paragon for string theory, then $\text{vdim } \overline{\mathcal{M}}_{g,0}(X, \beta) = 0$.

Example 1.8. For degree d rational curves in \mathbb{P}^N , we have $\int_{\beta} c_1(X) = \int_{d[\ell]} (N + 1)H = d(N + 1)$, and so $\text{vdim } \overline{\mathcal{M}}_{0,0}(\mathbb{P}^N, d) = (N + 1)(d + 1) - 4$. This agrees with the actual dimension ($N + 1$ polynomials of degree d), see footnote in Example 1.3.

Example 1.9. For a hypersurface X of degree D in \mathbb{P}^N , we have by the adjunction formula: $c_1(X) = -K_X = -(K_{\mathbb{P}^N} + [X])_X = (N + 1 - D)H|_X$, where H is the hyperplane class of \mathbb{P}^N . Hence $\text{vdim } \overline{\mathcal{M}}_{g,n}(X, d) = d(N + 1 - D) + (N - 4)(1 - g) + n$.

Similar to vdim , there will be a *virtual fundamental class*, of dimension vdim and coinciding with the usual fundamental class in case the theory is unobstructed. If $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is non-singular, the virtual fundamental class will turn out to be the Euler class (top Chern class) of the obstruction bundle² Ob :

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}} = e(\text{Ob}) \cap [\overline{\mathcal{M}}_{g,n}(X, \beta)]. \quad (1.10)$$

1.2 Gromov-Witten Invariants

1.2.1 Definition and Axioms

We proceed to define GW invariants for general target spaces X . These are invariant under complex structure deformations of X . The (primary) **Gromov-Witten invariants** are the following correlators involving cohomology classes $\gamma_i \in H^*(X, \mathbb{Q})$:

$$\langle \gamma_1 \cdots \gamma_n \rangle_{g, \beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{vir}}} \text{ev}_1^*(\gamma_1) \cup \cdots \cup \text{ev}_n^*(\gamma_n), \quad (1.11)$$

with the dimension matching condition $\sum \deg \gamma_i = 2 \text{vdim } \overline{\mathcal{M}}_{g,n}(X, \beta)$ (otherwise vanishing result), so that the result is a mere number (rational since our forms γ_i are \mathbb{Q} -valued).

The *enumerative meaning* of this number is as follows. The integrand is the cohomology class represented by those maps $(\Sigma_g, p_1, \dots, p_n, f)$ satisfying $f(p_i) \in Z_i \ \forall i$, where Z_i is a subvariety (of X) in the homology class dual to γ_i . Then the above GW invariant “counts” the number of curves (or rather maps) satisfying this requirement together with $f([\Sigma]) = \beta$. Because these curves usually come in infinite families, the final integral to compute will be an Euler number of a bundle parametrising these families, yielding a rational number.

We now list three properties of GW invariants [KM-94]:

²The *obstruction sheaf* is the pull-back of the tangent sheaf of X to $\overline{\mathcal{M}}_{g,n}(X, \beta)$ via (1.4), that is $R^1\pi_* \text{ev}_{n+1}^* T_X$, the fibre of which is $H^1(\pi^* \text{ev}_{n+1}^* T_X)$. If the moduli space is non-singular, it is locally free (i.e. a vector bundle).

(1) Fundamental class axiom: If one of the γ_i (say γ_n) is the fundamental class $[X] = 1_X \in H^0(X, \mathbb{Q})$, the GW invariant vanishes:

$$\langle \gamma_1 \dots \gamma_{n-1} [X] \rangle_{g, \beta} = 0,$$

since it can be rewritten with the same integrand, but with γ_n dropped and hence capped against $\overline{\mathcal{M}}_{g, n-1}(X, \beta)$; since this space has one dimension less, the cohomology class will not be of top degree anymore and the integral is zero. This holds when $n \geq 1$ (or $n + 2g \geq 4$ for $\beta = 0$).

(2) Divisor axiom: If γ_n is a 2-form with Poincaré dual $[Z_n]$ (divisor class), the candidate curves for stable maps will have their point p_n mapped on Z_n , and $\beta \cap [Z_n]$ will be a zero-chain with “number of points” $\int_{\beta} \gamma_n \in \mathbb{Q}$. This separate requirement in itself offers $\int_{\beta} \gamma_n$ “choices” of stable maps satisfying $f(p_i) \in Z_i$ ($i = 1, \dots, n-1$), so

$$\langle \gamma_1 \dots \gamma_n \rangle_{g, \beta} = \left(\int_{\beta} \gamma_n \right) \langle \gamma_1 \dots \gamma_{n-1} \rangle_{g, \beta}.$$

This holds when $n \geq 1$ (or $n + 2g \geq 4$ for $\beta = 0$).

(3) Constant map axiom: For $\beta = 0$, $g = 0$, the requirement $f(p_i) \in Z_i$ translates to $f(\Sigma) \in Z_1 \cap \dots \cap Z_n$. Since $[Z_1] \cap \dots \cap [Z_n] = \int_X \gamma_1 \cup \dots \cup \gamma_n$ we need $\sum \deg \gamma_i = 2 \dim X$. This agrees with $\sum \deg \gamma_i = 2 \text{vdim } \overline{\mathcal{M}}_{0, n}(X, 0)$ only if $n = 3$ (as $\overline{\mathcal{M}}_{0, 3} = \text{point}$). Hence $\langle \gamma_1 \dots \gamma_n \rangle_{0, 0} = 0$ except

$$\langle \gamma_1 \gamma_2 \gamma_3 \rangle_{0, 0} = \int_{\overline{\mathcal{M}}_{0, 3} \times X} \text{ev}^*(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3 \quad (1.12)$$

This does not hold for $g > 0$, as it would be an obstructed case (*i.e.* $\text{vdim} \neq \dim$), see also section 1.3.2.

1.2.2 Example: GW invariants of \mathbb{P}^1 :

This is an easy example. Note that $\text{vdim } \overline{\mathcal{M}}_{g, n}(\mathbb{P}^1, d) = 2d + 2g - 2 + n$. We only deal with two cohomology classes: the point class $[pt]$ generating $H^2(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$ and the fundamental class $[\mathbb{P}^1]$ generating $H^0(\mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}$.

For $\beta = 0$ (*i.e.* $d = 0$), we have to satisfy $\sum \deg \gamma_i = 2(2g - 2 + n)$, hence $g = 0, 1$ only. At $g = 0$ we have (by the constant map axiom) $n = 3$ and $\langle [pt][\mathbb{P}^1][\mathbb{P}^1] \rangle_{0, 0} = \int_{\mathbb{P}^1} [pt] = 1$. At $g = 1$ any n is allowed but all $\gamma_i = [pt]$, so (by the divisor axiom) $\langle [pt]^n \rangle_{1, 0} = \langle [pt] \rangle_{1, 0} = \int_{\overline{\mathcal{M}}_{1, 1}(\mathbb{P}^1, 0) \cong \mathbb{P}^1} [pt] = 1$.

For $\beta \neq 0$ (*i.e.* $d > 0$), we have to satisfy $\sum \deg \gamma_i = 2(2d + 2g - 2 + n)$; if $\gamma_i = [\mathbb{P}^1]$, the GW invariant vanishes by the fundamental class axiom. Hence all $\gamma_i = [pt]$; this can only hold if $2d + 2g - 2 = 0$, *i.e.* $d = 1$, $g = 0$: $\langle [pt]^n \rangle_{0, 1} = \langle [pt] \rangle_{0, 1} = \langle 1 \rangle_{0, 1} = \int_{\overline{\mathcal{M}}_{0, 0}(\mathbb{P}^1, 1)} 1 = 1$ because $[\overline{\mathcal{M}}_{0, 0}(\mathbb{P}^1, 1)]$ is a 0-cycle of degree 1 (only 1 point in that moduli space), or alternatively because $\langle [pt] \rangle_{0, 1} = \int_{\overline{\mathcal{M}}_{0, 1}(\mathbb{P}^1, 1) \cong \mathbb{P}^1} [pt] = 1$.

All other GW invariants vanish, in particular all those for $d > 1$ or $g > 1$. This is due to dimensional reasons and not because there are no maps of such degrees from such curves. In fact there are too many of them (infinite families) and our constraints $f(p_i) \in Z_i$ (a representative of the point class) are trivial – hence void. In order to end up with a finite number of maps, one needs rather to impose a stronger condition like specifying the branch points and their ramification indices. This is the *Hurwitz problem* and far from being solved.

1.2.3 Example: Rational Curves on the Quintic

Our next example concerns $g = 0$ invariants on CY threefolds (*i.e.* rational curves). For simplicity, we shall confine ourselves to the quintic hypersurface in \mathbb{P}^4 ($h^{1,1} = 1$), but most of our formulae are easily adapted to more general cases.

Note first that this example is unique in that $\text{vdim } \overline{\mathcal{M}}_{g,n}(X, \beta) = n$, allowing GW invariants with $n = 0$ since $\sum \deg \gamma_i = 0$ can be satisfied. That is, the absence of cohomology classes gives the GW invariant $\langle 1 \rangle_{g,d}$ the meaning of directly “counting” maps or, hopefully, curves in the quintic (without needing to pass through some points). The GW invariants at $g = 0$ are very simple and heavily restricted by $\sum \dim \gamma_i = 2n$.

For $\beta = 0$ they are $\langle \gamma_1 \gamma_2 \gamma_3 \rangle_{0,0} = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$; for the quintic with hyperplane class $H \in H^2(X, \mathbb{Z})$, line class $\ell = \frac{1}{5}H^2 \in H^4(X, \mathbb{Z}) = \mathbb{Z}$ and point class $[pt] = \frac{1}{5}H^3 \in H^6(X, \mathbb{Z}) = \mathbb{Z}$ this vanishes except for

$$\langle HHH \rangle_{0,0} = 5, \quad \langle H \ell [X] \rangle_{0,0} = \langle [pt][X][X] \rangle_{0,0} = 1, \quad \text{or} \quad \langle \gamma_1 \gamma_2 [X] \rangle_{0,0} \text{ for } \gamma_i \in H^3(X, \mathbb{Z}).$$

For $\beta \neq 0$, $\langle \gamma_1 \dots \gamma_{n-1} [X] \rangle_{0,d} = 0$ by the fundamental class axiom. The only other possibility is that all γ_i are 2-forms (H for the quintic): $\langle H \dots H \rangle_{0,d} = d^n \langle 1 \rangle_{0,d}$ by the divisor axiom. Hence all GW invariants are simply 0 or multiples of $\langle 1 \rangle_{0,d}$. (The same conclusion is true at $g > 0$ and for general CY threefolds.)

That $\text{vdim } \overline{\mathcal{M}}_{0,0}(X, \beta)$ is 0 suggests that there is a finite number of rational curves at each degree d (Clemens conjecture)³. This number $N_d := \langle 1 \rangle_{0,d}$ of rational maps of degree d coincides with the degree of the 0-cycle $[\overline{\mathcal{M}}_{0,0}(X, \beta)]^{\text{vir}}$. Brute force computation of the above Euler numbers leads to despair (was met with increasing complexity till degree 3); luckily Mirror Symmetry found a smart way of tackling all numbers in one stroke [CdGP-91]: $N_1 = 2875$, $N_2 = 4876875/8, \dots$

Multicovers and Instanton Numbers: Despite “counting” rational maps, these invariants are generally $\in \mathbb{Q}$ due to the orbifold structure of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ and of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, d)$. Moreover, they include contributions from k -fold covers of curves of degree d/k . An excess intersection calculation [AM-91] determines this contribution to be $1/k^3$ for each suc cover. Define now the *instanton numbers* n_d inductively by

$$N_d = \langle 1 \rangle_{0,d} =: \sum_{k|d} \frac{1}{k^3} n_{d/k}. \quad (1.13)$$

In our context, the n_d seem to play the role of numbers of smooth isolated rational curves of degree d in the quintic.⁴ A marvellous fact holds since Mirror Symmetry was able to generate all N_d in a row: *the n_d are all integers* !

It is those instanton numbers that we shall associate with the numbers of BPS states or the degeneracies of $sl(2, \mathbb{C})$ spin representations of such states wrapping bound states of D0- and D2-branes (see eqns (1.19), (1.26) and section 2.4).

We can even understand the “excess” factor of k^{-3} in (1.13). Had we rather computed $\langle H^3 \rangle_{0,d}$, we would have counted the number of degree d maps with the added constraint of three image points lying on three different hyperplanes. Again, contribution from multiple covers would have written this as $\sum_{k|d} (\text{number of degree } k \text{ curves}) \times (\text{number of } d/k\text{-fold covers of such curves})$, with the same constraint. A degree k curve intersects a hyperplane in k points, giving us k choices for the marking $f(p_i)$; we also have d/k choices for the pre-image p_i of the covering map, but since we have three markings, these choices are equivalent (by an automorphism of \mathbb{P}^1). Overall, we have $\langle H^3 \rangle_{0,d} = \sum_{k|d} n_k k^3$. On the other hand, this equals $d^3 \langle 1 \rangle_{0,d} = d^3 N_d$ by the divisor axiom, and we obtain our claim.

³Yet the actual dimension of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ is non-zero since for each such curve $f : \Sigma_0 \rightarrow X$, we have whole families of k -fold covers $f \circ g$ with $g : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree k .

⁴Indeed, for $d \leq 9$ (where the Clemens conjecture is known to hold), this can be verified; but for $d = 10$, problems arise due to double-covers of (6-nodal) degree 5 curves, and n_{10} is not quite the number of rational curves [V-93].

1.3 Gromov-Witten Potentials

From now on, we restrict ourselves to X being a CY threefold. The potential we give here is specifically derived in this context, but the logics and many of the formulae hold more generally. It relies on the convenient feature that $\langle 1 \rangle_{g,d}$ exists for CY threefolds and that non-zero GW invariants are of the form $\langle \gamma_1 \dots \gamma_n \rangle_{0,d} = d_1 \dots d_n \langle 1 \rangle_{g,d}$ (i.e. all γ_i must be 2-forms). The letter d now denotes the vector (of length $h^{1,1}(X)$) representing the class $\beta = \sum d_i \gamma^i$ in a basis $\gamma^1, \dots, \gamma^{h^{1,1}}$ of $H_2(X, \mathbb{Z})$.

1.3.1 Summing over Holomorphic Maps

Given a complexified Kähler class $\gamma = \sum t_i \gamma_i \in H^2(X, \mathbb{C})$, with $t_i \in \mathbb{C}$ and γ_i forming a basis of $H^2(X, \mathbb{Z})$, we would like to build functions $F_g(t_i, \bar{t}_i)$ of the complexified Kähler moduli (t_i, \bar{t}_i) using our space $\overline{\mathcal{M}}_{g,n}(X, \beta)$. However, this space carries a dependence on $\beta \in H_2(X, \mathbb{Z})$, and to obtain a dependence on (t_i, \bar{t}_i) only, we simply sum over all possible such classes β . It turns out that the correct procedure is to rather sum over all possible maps $f : \Sigma_g \rightarrow X$ whose images yield these classes. Of course the latter sum is much bigger since there are several maps f (perhaps infinite families) with image in one same class β . In fact, summing over all inequivalent maps to X (i.e. $\sum_{f: \Sigma_g \rightarrow X}$) amounts to sum over all $\beta \in H_2$ and weigh each term by the size of the moduli space of maps to β . This weighing factor is our basic Gromov-Witten invariant $\langle 1 \rangle_{g,\beta} := N_d^g := \deg [\overline{\mathcal{M}}_{g,0}(X, \beta)]^{\text{vir}}$. Thus we can replace $\sum_{f: \Sigma_g \rightarrow X} \dots$ by $\sum_{\beta} \langle 1 \rangle_{g,\beta} \dots$ or $\sum_d N_d^g \dots$.

We introduce the notation $\int_{\beta} \gamma = \beta \cap \gamma = \sum d_i t_i =: d \cdot t$ (here $d \in \mathbb{N}^{h^{1,1}}$ and $t \in \mathbb{C}^{h^{1,1}}$). We furthermore restrict the dependence of $F_g(t_i, \bar{t}_i)$ on the *holomorphic* Kähler moduli (t_i) , i.e. we demand the maps $f : \Sigma_g \rightarrow X$ to be holomorphic. We choose to sum over the exponential $e^{d \cdot t}$, and to guarantee a finite expression we require $\gamma \in$ complexified Kähler cone such that $d \cdot t < 0$. The function we now introduce is known as the genus g *Gromov-Witten potential* associated to the Kähler class $\gamma = \sum t_i \gamma_i$:

$$\begin{aligned} F_g(t_i) &:= \sum_{\substack{\text{hol maps} \\ f: \Sigma_g \rightarrow X}} \exp \int_{\Sigma_g} f^*(\gamma) = \sum_{\substack{\text{hol maps} \\ f: \Sigma_g \rightarrow X}} \exp \int_{\beta} \gamma \\ &= \sum_{\beta \in H_2} \langle 1 \rangle_{g,\beta} e^{\beta \cap \gamma} = \sum_d N_d^g e^{d \cdot t} = \\ &= \langle 1 \rangle_{g,0} + \sum_{\beta \neq 0} \langle 1 \rangle_{g,\beta} e^{\beta \cap \gamma} \end{aligned} \tag{1.14}$$

where further $e^{\beta \cap \gamma} = \sum_{n \geq 0} \frac{1}{n!} (\beta \cap \gamma)^n = \sum_{n \geq 0} \frac{1}{n!} \sum_{i_1 \dots i_n} t_{i_1} \dots t_{i_n} (\beta \cap \gamma_{i_1}) \dots (\beta \cap \gamma_{i_n})$, while $\langle 1 \rangle_{g,\beta} (\beta \cap \gamma_{i_1}) \dots (\beta \cap \gamma_{i_n}) = \langle \gamma_{i_1} \dots \gamma_{i_n} \rangle_{g,\beta}$ by the divisor axiom. Thus:

$$\begin{aligned} F_g(t_i) &= \langle 1 \rangle_{g,0} + \sum_{\beta \neq 0, n \geq 0} \frac{1}{n!} \sum_{i_1 \dots i_n} t_{i_1} \dots t_{i_n} \langle \gamma_{i_1} \dots \gamma_{i_n} \rangle_{g,\beta} \\ &= \langle 1 \rangle_{g,0} + \sum_{\beta \neq 0, n \geq 0} \frac{1}{n!} \langle \gamma^n \rangle_{g,\beta} \\ &= \langle 1 \rangle_{g,0} + \sum_{\beta \neq 0} \langle e^{\gamma} \rangle_{g,\beta} =: F_g^{\text{cl}} + F_g^{\text{qu}}, \end{aligned}$$

where we consider the contribution from constant maps as the *classical* potential, while all additions for $\beta \neq 0$ are seen as *quantum* corrections.

We group all potentials into one single *full GW potential*

$$F(t, \lambda) := \sum_{g \geq 0} F_g \lambda^{2g-2},$$

where λ is the string coupling constant and plays the role of formal expansion parameter.

For non-Calabi-Yau target space, the last set of equations are general enough to define F_g , but the γ_i 's span the whole of $H^*(X, \mathbb{C})$. So much for the general theory. Let us review some well-known contributions to the F_g .

1.3.2 Constant Map Contribution for CY Threefolds

The contribution from constant maps ($\beta = 0$) to the full GW potential is well-known [GeP-98] [KM-94]. It requires the *Hodge bundle* \mathbb{E} for the universal curve $\pi : \overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$: this is the rank g bundle on $\overline{\mathcal{M}}_{g,n}$ with fibre $H^0(\Sigma, \omega_\Sigma)$ over the point $(\Sigma, p_1, \dots, p_n)$.⁵ Denote its Chern classes by $\lambda_i := c_i(\mathbb{E})$.

For constant maps, the splitting $\overline{\mathcal{M}}_{g,n}(X, 0) = \overline{\mathcal{M}}_{g,n} \times X$ is accompanied by the splitting of the obstruction sheaf $\text{Ob} = \mathbb{E}^* \boxtimes T_X$. So from (1.10):

$$[\overline{\mathcal{M}}_{g,n}(X, 0)]^{\text{virt}} = [\overline{\mathcal{M}}_{g,n} \times X] \cap e(\mathbb{E}^* \boxtimes T_X).$$

With this split, the Euler character of the obstruction bundle is easy to calculate [GeP-98]:

$$e(\mathbb{E}^* \boxtimes T_X) = \frac{(-1)^g}{2} (c_3(X) - c_2(X) c_1(X)) \lambda_{g-1}^3, \quad g \geq 2,$$

and in the case of a Calabi-Yau threefold the expression reduces to $\frac{(-1)^g}{2} c_3(X) \lambda_{g-1}^3$. Due to the occurrence of $c_3(X)$, an Euler class, we cannot allow for γ_i 's to be integrated over X , so the only non-zero GW invariants for $\beta = 0$, $g \geq 2$ has $n = 0$.⁶

$$\langle 1 \rangle_{g,0} = \frac{(-1)^g}{2} \chi(X) \int_{\overline{\mathcal{M}}_{g,0}} \lambda_{g-1}^3 = (-1)^g \frac{\chi(X)}{2} \frac{B_{2g}}{2g(2g-2)!} \zeta(3-2g), \quad (1.15)$$

where B_{2g} are Bernoulli numbers⁷. The first step was reached in [BCOV-94] and the last step was computed in [FP-98]. Hence, if we formally set $\langle 1 \rangle_{0,0} := -\frac{\chi}{2} \zeta(3)$ and $\langle 1 \rangle_{1,0} := -\frac{\chi}{2} \zeta(1)/12$ for the genus 0 and 1 contributions, and using the expansion (1.27), we obtain the following constant map contributions to the full GW potential:

$$F_{\text{const}} := \sum_{g \geq 0} \lambda^{2g-2} \langle 1 \rangle_{g,0} = -\frac{\chi(X)}{2} \sum_{k=1} \frac{1}{k} \frac{1}{(2 \sin \frac{k\lambda}{2})^2} = -\frac{\chi(X)}{2} \sum_{n=1} \log(1 - e^{\pm i\lambda n})^n. \quad (1.16)$$

⁵That is, the g independent holomorphic differentials on the Riemann surfaces of genus g form vector spaces which patch together to form the Hodge bundle. Alternatively, $\mathbb{E} := \pi_* \omega_C$ where $C = \overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}$ is the universal curve and ω_C the relative dualising sheaf.

⁶The integral in $\int_{\overline{\mathcal{M}}_{g,0}} \lambda_{g-1}^3$ is our first (and only) example of *Hodge integrals*, i.e. integrals over $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{virt}}$ that include ψ and λ -classes:

$$\int_{[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{virt}}} \prod_{i=1}^n \psi_i^{a_i} \cup \text{ev}_i^*(\gamma_i) \cup \prod_{j=1}^g \lambda_j^{b_j}$$

where the $\gamma_i \in H^*(X, \mathbb{Q})$, i.e. not necessarily 2-forms, and the ψ_i are cotangent line classes over $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Without λ -classes we have *gravitational descendants*, and with no ψ -classes either we have *primary* GW invariants. A theorem of [FP-98] states that Hodge integrals over $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\text{virt}}$ can be uniquely reconstructed from the set of descendent integrals. In genus 0 or 1, the latter can even be expressed in terms of primary GW invariants [KM-94].

Note that in our case, the λ -class was hidden in the virtual fundamental class and not included as an extra parameter (as in genuine Hodge integrals over $\overline{\mathcal{M}}_{g,n}(X, \beta)$). Since the Hodge class λ_{g-1}^3 already matches the dimension of $\overline{\mathcal{M}}_{g,0}$, we cannot allow for extra classes; and anyway $n = 0$ means we have no ψ classes in the above constant term.

⁷Bernoulli numbers are defined by $\sum_0 B_k \frac{x^k}{k!} := \frac{x}{e^x - 1}$; they satisfy $B_{2k+1} = 0$ except $B_1 = -\frac{1}{2}$, and $B_{2k} = (-1)^{k+1} \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k) = -2k \zeta(1-2k)$.

This is the generating function for solid partitions (3d partitions), as proven originally in [M-1915] and related in this context to melting crystals by [ORV-03].

For the $\underline{g=0}$ case properly, we have $\mathbb{E} = 0$, so the obstruction bundle is trivial and the virtual fundamental class does not furnish us with a λ -class to integrate over $\overline{\mathcal{M}}_{0,n}$ (which is $n-3$ dimensional). Hence $e(\mathbb{E}^* \boxtimes T_X) = 1$ and $\langle 1 \rangle_{0,0} = 0$, and the only non-zero primary GW invariant occurs for $n=3$, as mentioned in (1.12): $\langle \gamma_1 \gamma_2 \gamma_3 \rangle_{0,0} = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$.

For $\underline{g=1}$, we have [GeP-98] $e(\mathbb{E}^* \boxtimes T_X) = c_3(X) - c_2(X)\lambda_1$, so the only non-zero primary GW invariant occurs for $n=1$:

$$\langle \gamma_i \rangle_{1,0} = [\overline{\mathcal{M}}_{1,1}(X, 0)]^{\text{virt}} = - \int_X c_2(X) \cup \gamma_i \int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = -\frac{1}{24} (c_2(X), \gamma_i) \quad (1.17)$$

Note that $\langle 1 \rangle_{1,0}$ would be acceptable from the dimensional point of view (expecting $\int_X c_3(X) \int_{\overline{\mathcal{M}}_{1,0}} 1$, similar to the $g=0, n=3$ term), but we rule this case out since it is unstable ($2g-2+n \leq 0$).

1.4 Multicovers and Known Cases of Excess Intersection

Having defined the GW invariants N_d^g , we now turn to the question of how a given image curve C in the CY threefold X contributes to them. This question will ultimately lead us to the physical approach of string theory.

1.4.1 Multicover Contributions

So far we have only mentioned the *degree* β (or d) of our maps, carefully avoiding the nature of the image curve. However, the latter also has a genus which we denote by r , and the Riemann-Hurwitz theorem tells us that a cover of such a curve can only come from a Riemann surface Σ_g with $g \geq r$. So N_d^g receives contributions from image curves C_r with genus r not greater than g . Moreover, the degree of the curve need not equal β , as N_d^g also has contributions from k -fold covers of curves of degree β/k if $\beta/k \in H_2(X, \mathbb{Z})$. Overall, we wonder what is the contribution to N_d^g from the map $f: \Sigma_g \rightarrow C_r$ of degree k . The additional distinction of the *genus* of the image curve is typical of the topological string theory approach, where this number has the physical meaning of the $SU(2)_L$ content of the spectrum of M2-branes wrapped on the curve (see below).

Let C be an image curve of degree $\beta \neq 0$ and arithmetic genus r . Define h by $g =: r + h$. Then C contributes to the genus g degree $k\beta$ basic GW invariant via $\overline{\mathcal{M}}_{g,0}(X, k\beta)$, *i.e.* via a k -fold cover of C (sometimes called degenerate contributions, though we shall keep this term for stricter cases). Let us denote C 's contribution by $C_r(h, k)$.⁸ Then N_d^g gets a contribution only from a curve C of degree d , but also from multicovers of curves of smaller degree and lower genus:

$$N_d^g =: \sum_{\substack{k|d \\ r \leq g}} C_r(g-r, k) n_{d/k}^r \quad (1.18)$$

where $k|d$ means $k|d_i \forall i$. This excess intersection formula is a generalisation of the previous one for rational curves (1.13). In this way of writing, $n_{d/k}^r$ takes the meaning of “number of holomorphic curves” of arithmetic genus r and degree d/k , since each such curve contributes an amount of $C_r(g-r, k)$. Equation (1.18) actually defines the $n_{d/k}^r$, and we suggest to call them the “virtual number of curves”. One expects them to be rational numbers for two reasons: first

⁸A good definition of this excess contribution, avoiding *rigidity* problems is the following:

$$C_r(h, k) := \int_{[\overline{\mathcal{M}}_{r+h,0}(C, k\beta)]^{\text{virt}}} e(R^\bullet \pi_* \text{ev}_1^* (\mathcal{O}_C \oplus K_C)[1]).$$

where K_C is the canonical sheaf.

because N_d^g and $C_r(h, k)$ are so; secondly because “counting” curves that are not isolated (occur in families as is often the case) involves integrals of Euler classes over such families and the latter Euler numbers are typically rational. Thus one is startled to discover that in all known examples, the n_d^r computed from known N_d^g and $C_r(g - r, k)$ turned out to be integers. This has remained a mystery ever since GW theory was developed.

We now introduce the so-called *BPS-invariants* \tilde{n}_d^r , which differ only slightly⁹ from the above n_d^r . They have their origin in the physical approach (see below) via the D-brane moduli space \mathcal{M} . If the latter is smooth, we can define them by

$$\tilde{n}_d^r := \int_{\mathcal{M}} e(T^* \mathcal{M}) = (-1)^{\dim \mathcal{M}} e(\mathcal{M}). \quad (1.19)$$

We will see shortly a more usual way to define them (see eqn (1.26)). It is useful to relate them to the N_d^g via a similar version of (1.18):

$$N_d^g =: \sum_{\substack{k|d \\ r \leq g}} \tilde{C}_r(g - r, k) \tilde{n}_{d/k}^r$$

which may be seen this time as defining the $\tilde{C}_r(h, k)$. We will see that the \tilde{n}_d^r are integers iff the n_d^r are so. The *Gopakumar-Vafa conjecture* states that the \tilde{n}_d^r are indeed integers.

1.4.2 Degree-Genus Relationship

One will expect the n_d^r to be non-zero only for specific combinations of r and d , since the arithmetic genus depends on the degree of the curve. For example, if the Calabi-Yau is a local¹⁰ \mathbb{P}^2 , *i.e.* the fibration $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$ with fibre the line bundle $\mathcal{O}(-3)$, then a curve of degree d can only sit in the plane and will have the usual arithmetic genus of $r = (d - 1)(d - 2)/2$. One can add δ nodes to the curve and thus lower the genus to obtain a geometric genus of $r - \delta$; this will contribute to the invariant $n_d^{r-\delta}$. Thus, from now on, we admit singular image curves and the superscript in n_d^r will actually denote the *geometric* genus. Another example is the local $\mathbb{P}^1 \times \mathbb{P}^1$ case, where a curve of bidegree (a, b) with respect to the two \mathbb{P}^1 's has a total degree $d = a + b$ and arithmetic genus $(a - 1)(b - 1)$.

A third example [KKV-99] of a local CY case, is given by choosing the base to be a del Pezzo surface E_n (*i.e.* the blow-up of \mathbb{P}^2 in n points): a curve in the base is described by multidegrees $d = (a; b_1 \dots b_n)$ corresponding to the class $aH - \sum b_i e_i$ (H is the hyperplane class and e_i are the exceptional divisors of the blow-ups in n points). Its genus is given by (see *e.g.* section 4.4.1 of [G-04])

$$r = \frac{(a - 1)(a - 2)}{2} - \sum_{i=1}^n \frac{b_i(b_i - 1)}{2}$$

and its overall degree is given by its intersection number with the anticanonical class $-K_{E_n} = 3H - \sum_{i=1}^n e_i$:

$$d = -K_{E_n} \cdot (a; b_1 \dots b_n) = 3a - \sum_{i=1}^n b_i,$$

⁹The only difference occurs at genus 1 ($\tilde{n}_d^1 = \sum_{k|d} n_k^1$, see section 1.5.7) and genus 2 ($\tilde{n}_d^2 = \sum_{k|d} \frac{d}{k} \tilde{n}_k^2$). These subtleties are rather scholastic and we encourage the reader to overlook them at a first reading.

¹⁰We call *local CY threefold* the total space X of a fibration $\mathcal{O}(K_B) \rightarrow B$ with fibre the canonical bundle of a surface B . This is non-compact but locally CY since $c_1(X) = c_1(\mathcal{O}(K_B) \otimes TB) = c_1(\mathcal{O}(K_B)) + c_1(B) = 0$. The line bundles $\mathcal{O}_{\mathbb{P}^d}(n) := \mathcal{O}_{\mathbb{P}^d}(nH)$, for $n \in \mathbb{Z}$ and H the divisor class of \mathbb{P}^d , are those whose transition functions are locally $(x_\alpha/x_\beta)^n$ and whose sections are the degree n homogeneous polynomials (also called *twisted sheaves of Serre* in [H-77]). Similarly, the base could rather be 1-dimensional and the fibre a rank two bundle, as in $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$.

where we have exploited $H^2 = 1$ and $e_i e_j = -\delta_{ij}$. We use this new integer d to label the n_d^r .

So these examples reflect the general pattern that one finds a curve of given genus only at sufficiently large degree. This is not to say that this curve will be no candidate for n_d^r with lower r : adding δ nodes to the curve lowers its geometric genus to $r - \delta$. In that way, singular curves are taken into account for the invariants $n_d^{r-\delta}$ for $\delta = 1, \dots, r$. So the numbers n_d^r only vanish if r is too large compared to d . Determining the maximal genus of a curve of given degree d in a particular space is the subject of Castelnuovo theory.

1.4.3 Known $C_r(h, k)$ and a Physical Assumption

The $C_r(h, k)$ have been computed for $r = 0, 1$:

For rational curves ($r = 0$), [FP-98] showed that for CY threefolds:

$$\sum_{h \geq 0} C_0(h, 1) \lambda^{2h} = \left(\frac{\sin \lambda/2}{\lambda/2} \right)^{-2}$$

$$C_0(h, k) = C_0(h, 1) \frac{1}{k^{3-2h}} = C_0(h, 1) \frac{1}{k^{3-2g}} \quad (1.20)$$

yielding in closed form:

$$C_0(h, k) = \begin{cases} k^{2g-3} |B_{2h}| / 2h(2h-2)! & h \geq 2 \\ 1/k^3 & h = 0 \\ 1/(12k) & h = 1 \end{cases}$$

The B_k are the most fashionable version of Bernoulli numbers, defined under (1.15). The cases $h = 0, 1$ were long known.

For elliptic curves ($r = 1$), the Riemann-Hurwitz formula allows for unbranched multicovers from other elliptic curves (*i.e.* $g = 1$, *i.e.* $h = 0$). [P-98](v.2 !) showed:

$$C_1(0, k) = \frac{\sigma_1(k)}{k} = \sum_{n|k} \frac{1}{n} \quad \text{and} \quad C_1(h, k) = 0 \quad (h > 0) \quad (1.21)$$

The last equation claims that elliptic curves only contribute to F_1 , although in principle the Riemann-Hurwitz formula admits branched covers from higher genus curves.¹¹

Generalising to higher genera, [P-98] proved:

$$\sum_{h \geq 0} C_r(h, 1) \lambda^{2h} = \left(\frac{\sin \lambda/2}{\lambda/2} \right)^{2r-2}. \quad (1.22)$$

We assume that the $\tilde{C}_r(h, k)$ enjoy the same properties, except for the first part of (1.21):

$$\tilde{C}_1(0, k) = \frac{1}{k} \quad (1.23)$$

With this small change done, we now would like to assume that (1.20) also holds at higher genera, *i.e.*

$$\tilde{C}_r(h, k) = \tilde{C}_r(h, 1) \frac{1}{k^{3-2(r+h)}} \quad (1.24)$$

¹¹In [BCOV-94] this was argued to be due to the fact that the moduli space $\overline{\mathcal{M}}_{1+h,0}(X, \beta)$ does not contain a factor of a torus. This absence of flat torus is crucial since factors of non-zero curvature in the action are needed to absorb the fermion zero modes.

which entails that $\tilde{C}_g(0, k) = k^{2g-3}$. Note that the $\tilde{C}_r(h, k)$ and the $C_r(h, k)$ only differ for $k > 1$. The assumption (1.24) cannot hold for the $C_r(h, k)$ themselves as it would suggest that multicovers of a genus g curve by another genus g curve yield a non-zero contribution, in contradiction to the Riemann-Hurwitz formula which prohibits such multicovers. In the D-brane or BPS picture, we face no such embarrassment, as D-branes are more than just curves; in fact they carry gauge bundles and are best treated as sheaves on the algebraic geometric side. In any case, (1.24) is rather an assumption cherished by physicists. We will see why this is a powerful assumption; it will enable us to derive now the expression by which the \tilde{n}_d^r are usually defined – eqn (1.26).

1.5 Explicit Forms of F and F_g for CY Threefolds

This section is devoted to algebraic manipulations of sums and products in order to arrive at elegant versions for the GW potentials on CY threefolds.

1.5.1 Evaluating F in Terms of \tilde{n}_d^r

We will combine all GW potentials F_g into a single generating function $F := \sum_0 \lambda^{2g-2} F_g$; then the latter result (1.22) and assumption (1.24) enter in the last step of the next calculation. For simplicity, we drop the constant map terms ($\beta = 0$) from the potentials and call the new potentials \tilde{F}_g and \tilde{F} . We also introduce $q^d := e^{t \cdot d}$.

$$\begin{aligned} \tilde{F}_g(t_i) &= \sum_{d>0} N_d^g q^d = \sum_{d, k|d, r \leq g} \tilde{C}_r(g-r, k) \tilde{n}_{d/k}^r q^d = \sum_{d, k>0, r \leq g} \tilde{C}_r(g-r, k) \tilde{n}_d^r q^{dk} \quad (1.25) \\ \tilde{F}(t_i) &= \sum_{g=0} \lambda^{2g-2} \tilde{F}_g = \sum_{\substack{r, h \geq 0 \\ d, k > 0}} \lambda^{2(r+h)-2} \tilde{C}_r(h, k) \tilde{n}_d^r q^{dk} \\ &= \sum_{r \geq 0, d > 0} \tilde{n}_d^r \lambda^{2r-2} \sum_{k > 0} q^{dk} \sum_{h \geq 0} \tilde{C}_r(h, k) \lambda^{2h} \\ &= \sum_{r \geq 0, d > 0} \tilde{n}_d^r \lambda^{2r-2} \sum_{k > 0} q^{dk} k^{2r-3} \sum_{h \geq 0} \tilde{C}_r(h, 1) (k\lambda)^{2h} \\ &= \sum_{r \geq 0, d > 0} \tilde{n}_d^r \lambda^{2r-2} \sum_{k > 0} \frac{1}{k} \left(\frac{\sin \frac{k\lambda}{2}}{\lambda/2} \right)^{2r-2} q^{dk} \end{aligned}$$

Hence

$$\tilde{F}(t_i) = \sum_{\substack{r \geq 0 \\ d, k > 0}} \tilde{n}_d^r \frac{1}{k} \left(2 \sin \frac{k\lambda}{2} \right)^{2r-2} q^{dk} \quad (1.26)$$

This last expression was obtained by the physical approach of [GV2-98] where the \tilde{n}_d^r had a more pertinent meaning (see section 2.4): the number of BPS states of charge d and $SU(2)_L$ content r in the M-theory compactification on the Calabi-Yau X . From a mathematical point of view, one may consider (1.26) as giving an alternative definition of the \tilde{n}_d^r , avoiding physical quantities.

1.5.2 Evaluating F_g in Terms of Polylogs $\text{Li}_{3-2g}(q^d)$

We shall now rewrite F_g in terms of polylogarithms and present some explicit examples of GW potentials or GW invariants like the local \mathbb{P}^1 case, an infinite product expression, or the 3-point

function. For convenience we recall the following expansions:

$$\left(2 \sin \frac{k\lambda}{2}\right)^{-2} = \frac{1}{(k\lambda)^2} + \frac{1}{12} + \frac{(k\lambda)^2}{240} + \dots + \frac{|B_{2n}|}{2n(2n-2)!} (k\lambda)^{2n-2} + \dots \quad (1.27)$$

$$\left(2 \sin \frac{k\lambda}{2}\right)^2 = (k\lambda)^2 - \frac{(k\lambda)^4}{12} + \dots + (-1)^{n+1} 2 \frac{(k\lambda)^{2n}}{(2n)!} + \dots$$

This allows us to express the \tilde{F}_g in terms of the \tilde{n}_d^r . For low genus we have:

$$\tilde{F}_0 = \sum_{d>0} \tilde{n}_d^0 \text{Li}_3(q^d) \quad \tilde{F}_1 = \sum_{d>0} \left(\frac{1}{12} \tilde{n}_d^0 + \tilde{n}_d^1 \right) \text{Li}_1(q^d) \quad (1.28)$$

$$\tilde{F}_2 = \sum_{d>0} \left(\frac{1}{240} \tilde{n}_d^0 + \tilde{n}_d^2 \right) \text{Li}_{-1}(q^d) \quad (1.29)$$

and in general ($g \geq 2$):

$$\tilde{F}_g(t_i) = \sum_{d>0} \left(\frac{|B_{2g}|}{2g(2g-2)!} \tilde{n}_d^0 + \frac{2(-1)^g}{(2g-2)!} \tilde{n}_d^2 + \dots + \tilde{n}_d^g \right) \text{Li}_{3-2g}(q^d) \quad (1.30)$$

The polylogarithms are defined by $\text{Li}_g(q) := \sum_1 \frac{q^n}{n^g}$ and satisfy

$$\text{Li}_1(q) = -\log(1-q), \quad \text{Li}_0(q) = \frac{q}{1-q}, \quad \text{Li}_{-1}(q) = \frac{q}{(1-q)^2}$$

and in general: $q\partial_q \text{Li}_g = \text{Li}_{g-1}$.

Of course these relations for \tilde{F}_g could have been directly obtained via (1.22), (1.24) and (1.25). In any case, the polylogarithms occur due to the multicovers, *i.e.* the sum over k . It should be emphasised that we could not replace \tilde{n}_d^r by n_d^r in \tilde{F} and \tilde{F}_g , as the assumption (1.24) would not hold for the algebro-geometric $C_r(h, k)$. The result (1.26) was first derived by [GV2-98] [KKV-99] from a physical approach (see section 2.3).

1.5.3 Example: the Local \mathbb{P}^1

An easy example of generating function for GW potentials is the local \mathbb{P}^1 case (*i.e.* non-compact CY space), the rank two concave bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. Here the conifold geometry forbids positive genus curves. The base \mathbb{P}^1 is the only rational curve, so all n_d^r vanish except $n_1^0 = 1$ (since $h^2(\mathbb{P}^1) = 1$, d is merely an integer). From (1.30) we then have:

$$\begin{aligned} \tilde{F}_{\mathbb{P}^1} &= \sum_{\geq 0} \lambda^{2g-2} \tilde{F}_g = \lambda^{-2} \text{Li}_3(q) + \sum_{g \geq 1} \lambda^{2g-2} \frac{|B_{2g}|}{2g(2g-2)!} \text{Li}_{3-2g}(q) \\ &= \sum_{l \geq 1} \frac{q^l}{l} \left((l\lambda)^{-2} + \sum_{g \geq 1} \frac{|B_{2g}|}{2g(2g-2)!} (l\lambda)^{2g-2} \right) \\ &= \sum_{l \geq 1} \frac{q^l}{l} \frac{1}{(2 \sin \frac{l\lambda}{2})^2} = \sum_{n \geq 1} \log(1 - e^{\pm i\lambda n} q)^n \end{aligned} \quad (1.31)$$

And adding the constant map contribution (1.16) with $\chi(\mathbb{P}^1) = 2$:

$$F_{\mathbb{P}^1} = \sum_{n \geq 1} \log[(1 - e^{\lambda n})^{-n} (1 - e^{\lambda n} q)^n],$$

where we take the freedom to treat the string coupling λ modulo $\pm i$.

1.5.4 Toda Equation for the Local \mathbb{P}^1

This is in fact not the full story. Physical considerations require to add some ‘classical terms’ to this ‘quantum’ piece. The classical part only occurs at genus 0 and 1 (*i.e.* powers λ^{-2} and λ^0) and is a polynomial in the modulus t :

$$F_{\text{cl}}(t) = \lambda^{-2} \left(-\zeta(3) + \frac{\pi^2}{6}t + i(m + \frac{1}{4})\pi t^2 - \frac{1}{12}t^3 \right) + \frac{1}{24}t$$

Under shifts of $t \rightarrow t \pm \lambda$, this classical part behaves as:

$$F_{\text{cl}}(t \pm \lambda) = F_{\text{cl}}(t) + \lambda^{-2} \left(\mp \frac{1}{4}t^2\lambda - \frac{1}{4}t\lambda^2 \pm \frac{1}{12}\lambda^3 + i(m + \frac{1}{4})\pi(\pm 2t\lambda + \lambda^2) \pm \frac{\pi^2}{6}\lambda \right) \pm \frac{\lambda}{24},$$

so that

$$F_{\text{cl}}(t + \lambda) + F_{\text{cl}}(t - \lambda) - 2F_{\text{cl}}(t) = -\frac{t}{2} + i(2m + \frac{1}{2})\pi.$$

Or in terms of the partition function $Z = e^F$:

$$\frac{Z_{\text{cl}}(t + \lambda)Z_{\text{cl}}(t - \lambda)}{Z_{\text{cl}}(t)^2} = e^{-\frac{t}{2} + i\frac{\pi}{2}}.$$

Why are we particularly interested in this last combination ? Well, the left hand side is just the homogeneous part of the Toda equation; and we now would like to see how the ‘quantum’ piece of our local \mathbb{P}^1 GW potential behaves. From the above product form of $F_{\mathbb{P}^1}$ and using $e^{\lambda n}e^{t \pm \lambda} = e^{\lambda(n+1)}q$, we readily obtain:

$$Z_{\text{qu}}(t + \lambda) = Z_{\text{qu}}(t) \prod_{n=1} (1 - e^{n\lambda}q)^{-1}, \quad Z_{\text{qu}}(t - \lambda) = Z_{\text{qu}}(t) \prod_{n=0} (1 - e^{n\lambda}q),$$

so that

$$\frac{Z_{\text{qu}}(t + \lambda)Z_{\text{qu}}(t - \lambda)}{Z_{\text{qu}}(t)^2} = (1 - q),$$

and the similar expression for the full (classical + quantum) change in the GW partition function of the local \mathbb{P}^1 reads:

$$\Delta Z = q^{-1/2}i(1 - q) = -2i \sinh \frac{t}{2}.$$

Thus the partition function of the local \mathbb{P}^1 satisfies the homogeneous Toda equation up to this inhomogeneity.

1.5.5 The Product Expression

If we had not restricted to the local \mathbb{P}^1 case and were interested in the part of the GW potential coming only from constant maps and rational curves ($r = 0$) on a general CY threefold, we would simply put back the \tilde{n}_d^0 in the above (1.31) and obtain:

$$\begin{aligned} F_{D0-D2} &= \sum_{n \geq 1} \log \left[(1 - e^{\lambda n})^{-n \frac{\chi}{2}} \prod_{d > 0} (1 - e^{\lambda n} q^d)^{n \tilde{n}_d^0} \right] \\ &= \log \prod_{\substack{n \geq 1 \\ d \geq 0}} (1 - e^{\lambda n} q^d)^{n \tilde{n}_d^0} \end{aligned}$$

setting formally $\tilde{n}_0^0 := -\frac{\chi}{2}$. This is just the Gopakumar-Vafa result [GV1-98] for contributions from constant plus rational maps (*i.e.* D0- and D2-branes).

In the same vein, we can rewrite the full GW potential (1.26) in product form by Fourier expanding the sine term:

$$(2 \sin x/2)^{2r-2} = \sum_{|j| \leq r-1} (-1)^j \binom{2r-2}{r-1+j} e^{ijx} =: \sum_{|j| \leq r-1} c_{r,j} e^{ijx} \quad \text{for } r \neq 0, 1,$$

while the sum is infinite in the $r = 0$ case and $c_{0,j} = -j$: $(2 \sin x/2)^{-2} = -\sum_{j \geq 1} j e^{\pm ijx}$. Then

$$\begin{aligned} \tilde{F} &= \sum_{\substack{r \geq 0 \\ d, k > 0}} \tilde{n}_d^r \frac{1}{k} \left(2 \sin \frac{k\lambda}{2} \right)^{2r-2} q^{dk} = \sum_{\substack{d > 0 \\ j \in \mathbb{Z}}} \left(-\sum_{r \geq 0} c_{r,j} \tilde{n}_d^r \right) \log(1 - e^{i\lambda j} q^d) \\ &= \log \prod_{\substack{d > 0 \\ j \in \mathbb{Z}}} (1 - e^{i\lambda j} q^d)^{M_{d,j}}, \end{aligned}$$

where we have set $M_{d,j} := -\sum_{r \geq 0} c_{r,j} \tilde{n}_d^r$. In subsection 1.4.2, we argued that the \tilde{n}_d^r vanish if r is too large compared to d ; that is, for fixed d , there are only a finite number of r to take into account, say $r < r_0(d)$. This is why the sum for the $M_{d,j}$ is a finite one. Moreover, except at $r = 0$, the sum over k is finite, so for fixed d one could write $k < k_0(d)$; but the $r = 0$ case makes the sum over j actually infinite.

As in the local \mathbb{P}^1 case, we can incorporate the constant map contribution (1.16) with $M_{0,j} = -j \frac{\chi}{2}$:

$$F = \log \prod_{\substack{d \geq 0 \\ j \in \mathbb{Z}}} (1 - e^{i\lambda j} q^d)^{M_{d,j}}.$$

The motivation behind this alternative expression for the GW potential will appear later when we shall try to relate this product to a modular product à la Borcherds. Such infinite products exhibit interesting symmetries, including automorphic properties.

Unlike the local \mathbb{P}^1 case, this very general product does not satisfy the homogeneous Toda equation up to a small inhomogeneity. This is due to the complicated nature of the powers $M_{d,j}$.

1.5.6 The Three-Point Function

To conclude this subsection, we give the Yukawa coupling or three-point function, which is the A-model correlation function for the quintic threefold in the context of Mirror Symmetry. Since $h^{1,1} = 1$ for this CY manifold, there is only one Kähler modulus t_1 , one Kähler class $\gamma_1 = H$ (the hyperplane class), and the degree d of the image curve is simply an integer :

$$\begin{aligned} \langle \gamma_1 \gamma_1 \gamma_1 \rangle_0 &:= \sum_{\beta} \langle \gamma_1^3 \rangle_{0,\beta} q^\beta = \sum_{d > 0} N_d d^3 q^d \\ &= \partial_{t_1}^3 \tilde{F}_0 =: \Phi_{111} \\ &= \sum_{d > 0} \tilde{n}_d^0 d^3 \text{Li}_0(q^d) = \sum_d \left(\sum_{k|d} k^3 \tilde{n}_k^0 \right) q^d \end{aligned}$$

where in the third step we have used the divisor axiom for GW invariants: $\langle \gamma_1^3 \rangle_{0,\beta} = \langle 1 \rangle_{0,\beta} \left(\int_{\beta} \gamma_1 \right)^3 = N_d d^3$. From this – or directly from the expression for \tilde{F}_0 – we obtain the genus 0 GW invariant (1.13): $N_d = \sum_{k|d} k^{-3} \tilde{n}_{d/k}^0$.

1.5.7 Subtleties Around n_d^r and \tilde{n}_d^r

In [BCOV-94], the contribution from genus 2 curves to \tilde{F}_2 was found to be $\sum_d n_d^2 q^d$, while (1.29) purports $\sum_d n_d^2 \text{Li}_{-1}(q^d) = \sum_{d,k} \tilde{n}_d^2 k q^{dk}$, suggesting that $n_d^2 = \sum_{k|d} \frac{d}{k} \tilde{n}_k^2$. This demonstrates that the \tilde{n}_d^r in general do not have the meaning of ‘counting curves’. In fact, they rather count BPS states, and the extra sum over k stands for multicovers of embedded D-branes. For instance, [MM-98] have taken [BCOV-94]’s result for \tilde{F}_2 when extracting the number of genus 2 curves from a similar expression derived by entirely different methods (see heterotic treatment below).

The expression (1.26) reflects also the fact that BPS states at genus 1 do not have any bubblings to higher genera, *i.e.* \tilde{n}_d^1 only appears in \tilde{F}_1 (since $r = 1$ implies zeroth power of λ), see also (1.30). From the polylog expression (1.28) for \tilde{F}_1 , we recover

$$N_d^1 = \frac{1}{d} \sum_{k|d} \left(\frac{1}{12} k \tilde{n}_k^0 + k \tilde{n}_k^1 \right) = \sum_{k|d} \left(\frac{1}{12} \frac{1}{k} \tilde{n}_{d/k}^0 + \frac{1}{k} \tilde{n}_{d/k}^1 \right)$$

and (of course) the second part of (1.21) and (1.23).

If we opt to express \tilde{F}_1 in terms of n_d^1 rather than \tilde{n}_d^1 , the choice $\tilde{n}_d^1 = \sum_{k|d} n_k^1$ will prove propitious:

$$\tilde{F}_1 = \sum_{d>0} N_d^1 q^d = \sum_{d>0} \left(\frac{1}{12} \tilde{n}_d^0 \text{Li}_1(q^d) - n_d^1 \log P(q^d) \right),$$

which is the form appearing in [BCOV-93] (with n_d^1 counting *curves*), wherein $P(x) := \prod (1 - x^n)$ and

$$-\log P(q^d) = \sum_{n=1} \text{Li}_1(q^{dn}) = \text{Li}_1 \left(\frac{1}{1 - q^d} \right) = \sum_{n,k \geq 1} \frac{q^{dnk}}{k}.$$

Whence the expression for the genus 1 basic GW invariant in terms of (geometrical) instanton invariants:

$$N_d^1 = \frac{1}{d} \sum_{k|d} \left(\frac{1}{12} k n_k^0 + \sigma\left(\frac{d}{k}\right) k n_k^1 \right)$$

with $\sigma(n) := \sum_{k|n} k$ and with no difference between \tilde{n}_d^0 and n_d^0 .

So the difference between the physical invariants \tilde{n}_d^1 and the geometrical invariants n_k^1 is that when using the latter, we have an extra \sum_n in \tilde{F}_1 . From the M-theory perspective, this is due to bound states of n M2-branes wrapped on the torus Σ_1 , being indecomposable stable $U(n)$ bundles over the torus. This is in addition to the possibility for a single M2-brane to wrap the torus in a k -fold way. These bound states solve a puzzle that occurred on the last page of [GV2-98], where the M-theory approach seemed to disagree with previous results from physics and geometry [BCOV-93]. Note that at other genera than $r = 1, 2$, there appears to be no difference in the invariants.

2 String Theory Approach

This section is entirely physical in substance and contains no computation. It presents the link from GW theory to topological and type IIA string theories and the meaning of the GW invariants in the latter context (then denoted \tilde{n}_d^r). They are degeneracies of bound states of D0 and D2-branes for given charge and spin of the BPS state. Chapter 3 will complete the physical picture by describing another context in which GW potentials can be computed: heterotic strings.

2.1 Topological Strings

Let us now recall how M-theory arrives at the above results and what meaning it gives to the variables. $N = 2$ SCFT's are the building blocks for string theories. In particular, we obtain a topological theory (A model, resp. B model) by twisting the fermion numbers of the CFT (Kähler twisting, resp. complex twisting). Moreover, after coupling it to gravity (*i.e.* after insertion of $3g - 3$ Beltrami operators), the theory allows for a partition function at genus g , denoted by F_g . In case of the weak coupling limit of the A model, the latter superpotential term should be a *holomorphic* function of the Kähler moduli t^i only, since it involves only chiral superfields (so-called F-terms). Hence this limit is obtained by letting the anti-holomorphic moduli tend to infinity, $\bar{t} \rightarrow \infty$, also known as the holomorphic limit or topological limit. Most interesting cases are susy sigma models with target space a Kähler manifold (which we will take to be a Calabi-Yau X , for convenience): they yield an $N = 2$ QFT whose only finite terms in the action – in the above limit – are those for which $\bar{\partial}X_i = 0$, *i.e.* the bosonic coordinates on the target space should be holomorphic functions. From the close analogy between an $N = 2$ SCFT and the bosonic string theory, we are led to introduce Riemann surfaces Σ_g and interpret the bosonic coordinates X_i to describe holomorphic maps from Σ_g to the Calabi-Yau X . Overall, we re-interpret the topological partition function F_g as counting holomorphic maps, or what amounts to the same, their image, *i.e.* holomorphic curves embedded in X .

2.2 Link with Type IIA Theory

The question arises as to which string theory has a similar description? From what we already said about the topological partition function F_g and also recalling that it is left-right symmetric as well as related to twisting of a susy sigma model for closed string theory, we are led to consider amplitudes with $2g - 2$ RR fields of charge $3/2$ on a Riemann surface of genus g . Incidentally, type IIA superstrings compactified on a CY threefold yields an $N = 2$ theory, and $2g - 2$ graviphoton vertices just satisfy the requirements. Indeed, it can be verified that its low-energy effective action coupled to supergravity contains the following Lagrangian F-terms:

$$\int F_g R_+^2 F_+^{2g-2} \quad (2.1)$$

where the insertions of the graviphoton fields F_+ account for the change from ordinary type IIA (*i.e.* non-twisted) theory to a properly twisted theory. R_+ denotes the self-dual part of the Riemann curvature of the compactification manifold X (*i.e.* insertion of two gravitons). These F-terms only appear at genus g (see section 3.1) and do not receive other quantum corrections (\hbar corrections), not even at non-perturbative level, as the dilaton of type IIA belongs to a hypermultiplet while $N=2$ SUGRA forbids adpendance of the F_g on matter hypermultiplets. Therefore the tree-level prepotential F_0 (whose Kähler parameters t_i are in vector multiplets) is *exact* at the full quantum level. However, it will suffer worldsheet instanton corrections (α' corrections, or rather α'/R^2 corrections where R is the Kähler parameter – or radius – of the compactification space), so a neat trick is to compute it via mirror symmetry, where Kähler and complex moduli are swapped. In any case, topological string theory offers a much easier way to compute the above F-terms.

Twisted type IIA compactified on the Calabi-Yau X re-interprets topological amplitudes (*i.e.* counting holomorphic curves) as corrections to $R_+^2 F_+^{2g-2}$. That is, F_g is understood as the coupling of the action terms $R_+^2 F_+^{2g-2}$. When viewing X as a $K3$ -fibration over the base \mathbb{P}^1 , the weak coupling limit is the large volume limit, say large volume of the base. Since these amplitudes are exact beyond genus g (free of quantum corrections), we might just as well compute them in the regime of strong coupling constant, where M-theory comes out of

hiding. At any rate, topological string theory computes exact quantities of the physical type IIA string theory.

2.3 M-Theory and the Schwinger Computation

The trick we shall use consists in integrating out light BPS states. Alas, in the large CY limit, D-branes grow in size and thus in mass. So we are looking for an additional limit, overriding the first one, where D0 and D2-branes become the lightest states. This is where M-theory enters the game: since M-theory compactified on a circle S^1 of radius R_{10} tends to type IIA as $R_{10} \rightarrow 0$ and since $g_s = 2\pi R_{10}$, strong coupling (large g_s) is but the decompactification limit of IIA where one recovers the full 11d M-theory. The advantage of this limit is that the relevant (*ie* lightest) BPS states are then D0 branes and D2 branes of IIA string theory, *ie* in terms of M-theory: KK modes and M2 branes respectively, or bound states thereof.

When M-theory is additionally compactified on the Calabi-Yau X , the large volume of the latter gives us back the perturbative regime of type IIA, where the F_g are the topological string partition functions and are given by the worldsheet instanton sum (1.14); that is, the images of the holomorphic maps $\Sigma_g \rightarrow X$ are just the supersymmetric cycles on which wrap the M2 branes of our M-theory. Note that the two compactification limits of large circle and large CY have an opposite effect on the string coupling: the first increasing it, the second lowering it. We choose the limits such that the first effect be the dominant one, *ie* strong coupling, where the lightest states are the D0- and D2-branes.

Note also that our strategy of integrating out light states has close analogy with Seiberg-Witten theory, where light magnetic monopoles become massless at singular points of the moduli space. These generate monodromies for the scalar Higgs field ($a \sim \langle \phi \rangle$) and its dual ($\hat{a} = \partial\mathcal{F}/\partial a$), and are integrated out via a one-loop integral. This integral is the only field theory contribution. There is no tree-level contribution, as SW-theory is a free theory near a singularity (*ie* where dyons or monopoles become massless). It is also exact at one-loop, so there is no question of higher loop contributions. The same applies to the Schwinger one-loop calculation below, *ie* it captures both perturbative and non-perturbative parts: the one-loop calculation is the exact result.

The variables occurring in \tilde{F} of (1.26) have the following physical meaning: The graviphoton field strength F_+ has been absorbed into the string coupling constant g_s to form the parameter $\lambda = g_s F_+$ in $F = \sum_0 \lambda^{2g-2} F_g$. The quantised momentum of a BPS state around the compactification circle S^1 is labelled by an integer n , which is summed over all the spectrum and turned into the integer k of (1.26) after Poisson resummation. The corrections (2.1) are calculated in [GV1-98] via a one-loop Schwinger integral with BPS states running around the loop, corresponding to bound states of D2-branes with n D0-branes. The latter have a charge (=mass) $e = m = |Z|$ with $Z = (A + 2\pi i n)/g_s$ where $n \in \mathbb{Z}$ is their momentum quantum number and A is their area: $A = 0$ for D0-branes (these account for the constant map contribution), or $A = d \cdot t$ for D2-branes (these account for the contribution from rational curves in the class d). These D2-branes are assumed to have the topology of S^2 , but no spin. The one-loop Schwinger integral computes the free energy of a charged scalar (charge e , mass m) in a constant self-dual field F_+ :

$$\begin{aligned} F(\lambda) &= \int_{\epsilon}^{\infty} \frac{ds}{s} \operatorname{tr} e^{-s(\Delta+m^2)} \\ &= \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{e^{-sZ/F_+}}{(2 \sinh s/2)^2} \end{aligned} \tag{2.2}$$

which we sum over all momenta $n \in \mathbb{Z}, n \neq 0$, and Poisson resum via $\sum_{n \in \mathbb{Z}} e^{inx} = \sum_{k \in \mathbb{Z}} \delta(x -$

$2\pi k$) to obtain:

$$F(\lambda) = \sum_{k \geq 1} \frac{1}{k} \frac{q^{-dk}}{(2 \sinh \frac{k g_s F_+}{2})^2}$$

This yields (1.26) upon taking all degrees into account weighed by \tilde{n}_d^r , redefining the topological string coupling constant $\lambda = g_s F_+ \rightarrow i\lambda$, swapping the Kähler cone ($t_i \rightarrow -t_i$), as well as incorporating the spin of the BPS state; mysteriously enough, the contributions from higher genus curves are obtained not by explicitly changing the topology of D2-branes but by allowing them to have spin: $\text{tr } e^{-2k\lambda J_3} \sim (2 \sin k\lambda/2)^{2r}$ for spin content $I_1 \otimes I_r$ (see section 2.4). This accounts for the extra sine term in (1.26) and the necessary r -dependence. Maybe one could invoke the 4d gravitino $\psi^\alpha = \psi_\mu^\alpha dx^\mu$ to account for this r -dependence, as there are r holomorphic one-forms dx^μ on a curve (=brane) of genus r .

2.4 Physical Meaning of the \tilde{n}_d^r

The invariants \tilde{n}_d^r already introduced in eqns (1.19) and (1.26) have the following interpretation: The BPS states are equivalent, in the M-theory perspective, to M2 branes wrapping susy cycles in a given homology class d . The latter class also describes the mass (or charge, or tension) of the BPS states – *i.e.* the area $d \cdot t$ of the M2 brane – in terms of the Kähler moduli t_i of the Calabi-Yau. An additional label is the transformation property under the spatial Lorentz group $SO(4) = SU(2)_L \times SU(2)_R$ in 4 + 1 dimensions, given by two half-integers (j_L, j_R) . If we denote by N_{j_L, j_R}^d the number of BPS states with these quantum numbers, then the latter transform in the following representation:

$$[(\frac{1}{2}, 0) \oplus 2(0, 0)] \otimes \bigoplus_{j_L, j_R} N_{j_L, j_R}^d [(j_L, j_R)].$$

The left-moving content, $[(\frac{1}{2}) + 2(0)] \otimes [j_L]$, is the usual N=2 BPS multiplet for a spin j ground state. We shall make use of the basis $I_r := [(\frac{1}{2}) + 2(0)]^{\otimes r}$, or explicitly:

$$\begin{aligned} I_0 &= (0) \\ I_1 &= 2(0) \oplus (\frac{1}{2}) \\ I_2 &= 5(0) \oplus 4(\frac{1}{2}) \oplus (1) \\ I_3 &= 14(0) \oplus 14(\frac{1}{2}) \oplus 6(1) \oplus (\frac{3}{2}) \\ &\dots \end{aligned}$$

Since F_+ only couples to the left spin quantum numbers, only the degeneracy of the left spin content will be an invariant of the BPS spectrum. That is, we need to sum over the right representation and, when expressing the result in a suitable basis, we obtain our sought-for invariants \tilde{n}_d^r of the theory:

$$\bigoplus_{j_L} \left(\sum_{j_R} N_{j_L, j_R}^d (-1)^{2j_R} (2j_R + 1) \right) [(j_L)] =: \bigoplus_r \tilde{n}_d^r I_r$$

where $(2j_R + 1)$ is the right spin degeneracy, and $(-1)^{2j_R}$ accounts for bose/fermi statistics. This defines the \tilde{n}_d^r in terms of the basis I_r .

The merit of the basis I_r is that $\text{tr}_{I_r} \dots = (\text{tr}_{I_1} \dots)^r$, so that a trace over the above representation content yields

$$\text{tr}_{\sum_r \tilde{n}_d^r I_r} (-1)^{2J_3} y^{2J_3} = \sum_r \tilde{n}_d^r (\text{tr}_{I_1} (-1)^{2J_3} y^{2J_3})^r = \sum_r \tilde{n}_d^r (2 - y - y^{-1})^r.$$

We have placed ourselves in the context of N=2 compactifications, where y keeps track of the third component of the $SU(2)$ current J (*i.e.* of the $U(1)$ subgroup).

For instance, if we have four left-moving bosonic oscillators α_{-n}^i with $SU(2)_L \times SU(2)_R$ (space-time) content $(\frac{1}{2}, \frac{1}{2})$, their partition function is

$$\prod_{n \geq 1} \frac{1}{(1 - yq^n)^2 (1 - y^{-1}q^n)^2} = \sum_{d \geq 0} \left(\sum_{r=-d}^d c_d^r y^r \right) q^d = \sum_{d \geq 0} \left(\sum_{r=0}^d \tilde{n}_d^r (2 - y - y^{-1})^r \right) q^d = \text{tr} (-1)^{2J_3} y^{2J_3} q^{L_0}$$

with the trace again over $\sum_r \tilde{n}_d^r I_r$. Concretely, at level $d = 3$, we have $\tilde{n}_3^r = 40, -60, 28, -4$ for $r = 0, 1, 2, 3$. Of course, this coincides with taking all combinations of oscillators at level 3, namely $\alpha_{-1}^i \alpha_{-1}^j \alpha_{-1}^k |0\rangle$, $\alpha_{-1}^i \alpha_{-2}^j |0\rangle$, $\alpha_{-3}^i |0\rangle$, and summing over the right spin content and weighing each term by $(-1)^{2j_R} (2j_R + 1)$:

$$\left[\left(\frac{3}{2}, \frac{3}{2} \right) \oplus \left(\frac{1}{2}, \frac{1}{2} \right) \right] \oplus \left[(1, 1) \oplus (1, 0) \oplus (0, 1) \oplus (0, 0) \right] \oplus \left[\left(\frac{1}{2}, \frac{1}{2} \right) \right] \longrightarrow 4(0) - 4\left(\frac{1}{2}\right) + 4(1) - 4\left(\frac{3}{2}\right) \\ = 40I_0 - 60I_1 + 28I_2 - 4I_3,$$

where on the *lhs* the three square brackets refer to the content of the three combinations of oscillators. Note that for operators of different level, the tensor product distributes into the brackets: $(\frac{1}{2}, \frac{1}{2})_{-1} \otimes (\frac{1}{2}, \frac{1}{2})_{-2} = (\frac{1}{2} \otimes \frac{1}{2}, \frac{1}{2} \otimes \frac{1}{2}) = (1 \oplus 0, 1 \oplus 0)$.

Another instance is a heterotic compactification on $K3 \times T^2$, where the 24 left oscillators α_{-n}^i with $SU(2)_L \times SU(2)_R$ content $20(0, 0) \oplus (\frac{1}{2}, \frac{1}{2})$ and partition function

$$\prod_{n \geq 1} \frac{1}{(1 - q^n)^{20} (1 - yq^n)^2 (1 - y^{-1}q^n)^2} = \sum_{d \geq 0} \left(\sum_{r=0}^d \tilde{n}_d^r (2 - y - y^{-1})^r \right) q^d, \quad (2.3)$$

which is just the p^0 term of equation (6.3) of [KKV-99]; that is, the \tilde{n}_d^r have here the enumerative meaning of counting BPS states wrapped on $K3$.

By construction, the \tilde{n}_d^r are integers, as they merely count BPS states. That these integer quantities are the same as those appearing in the Gromov-Witten potential \tilde{F} is a bold claim, given that our original topological invariants were only expected to be rational numbers: the counting of holomorphic curves from a Riemann surface to a threefold will in general involve integrals over moduli spaces of holomorphic maps, *i.e.* top characteristic classes or Euler numbers. So we see that M-theory offers a novel approach to the Gromov-Witten invariants.

3 Relation with Heterotic Strings

We now head towards topics related to the GW potential, both physical and mathematical. This small section is, much like section 2, devoted to understanding the stringy background of GW theory and its genus g potential. It is purely physical, without computations, and presents the one-loop integral of the heterotic string amplitude and an attempt at solving it via Jacobi forms.

The M-theory (or type IIA) approach to Gromov-Witten invariants is the most recent one, but another attempt from string theory proved fruitful: this is the calculation in heterotic string theory, following the discovery in 1995 of its description dual to the type IIA theory. Given that the perturbative expansion is governed by the dilaton, that the dilaton lies in a hypermultiplet in type IIA and in a vector multiplet in heterotic, and that the two kinds of multiplets do not mix with each other, the duality then allows us to extract non-perturbative knowledge (instanton corrections,...) in one model from perturbative expansions in the dual model. The similar properties of the topological couplings F_g in heterotic theory compactified on $K3 \times T^2$ and IIA compactified on a CY threefold were seen as a test of this $N = 2$ duality. In particular the F_g satisfy the same holomorphic anomaly equation in the respective weak coupling limits (*i.e.* $S \rightarrow \infty$ and $t \rightarrow \infty$).

3.1 Moduli and the One-Loop Level

In the semi-classical limit (*i.e.* taking the dilaton $S \rightarrow \infty$), heterotic theory – with embedding of the spin connection in the gauge group – is described by two complex moduli $(T, U) =: y$, element of the Narain moduli space $\mathcal{N}^{2,2} = O(2, 2)/O(2) \times O(2) \simeq \mathcal{H} \times \mathcal{H}$. These are the moduli corresponding to the compactification on T^2 and governing its bosonic partition function. Since the string coupling equals $g_s = e^{iS}$, the above limit is the weak coupling limit. This coincides with the topological limit on the type IIA side, since one of the Kähler moduli of the Calabi-Yau is given by S itself. For instance, in the case of the hypersurface of degree 24 in $\mathbb{P}(1, 1, 2, 8, 12)$ (a $K3$ fibration), the precise map between the heterotic moduli and the IIA Kähler moduli is $t = (U, S, T - U)$, and sending $S \rightarrow \infty$ means infinite volume of X – or rather of the base \mathbb{P}^1 of the $K3$ -fibration.

An important difference between the IIA and heterotic calculations is that in the former the F_g are generated at g -loop level, while in the latter they all occur at 1-loop level (due to $N = 2$ non-renormalisation theorems)¹². This can be argued as follows: the F_g 's have the meaning of moduli-dependent couplings for the low-energy effective action terms $R^2 F_+^{2g-2}$. They are homogeneous of degree $2 - 2g$ in the superfields X^I , whose scalar component (X^0) has the following dependence on the string coupling and the Kähler potential:

$$F_g(X) = (X^0)^{2-2g} F_g(Z) = \left(\frac{e^{K/2}}{g_s} \right)^{2-2g} F_g(Z)$$

where $Z = X/X^0$ are the moduli. The Kähler potential K depends on the string coupling g_s via the dilaton: this dependence is nil in the case of type IIA (as the dilaton is in a hypermultiplet), and $\log g_s^2$ in the case of heterotic strings. Accordingly, $F_g(X)$ is of order g_s^{2g-2} or g_s^0 respectively in the string coupling. Counting string loops, this means g -loop or one-loop respectively.

In both theories, the prepotential develops logarithmic singularities reminiscent of the Seiberg-Witten analysis: different branches of the enhanced symmetry locus (ESL) collapse in the large moduli limit. On the IIA side, this corresponds to the conifold locus in the moduli space of Calabi-Yau 3-folds. On the heterotic side, it corresponds to codimension 1 surfaces of the moduli space where the gauge group $U(1)^{n_v+2}$ is enhanced to $SU(2)$ because two vector multiplets become massless. Note that the rank of the gauge group is the number n_v of vector multiplets (containing the compactification moduli) plus two extra vectors (graviphoton from the sugra multiplet and the vector in the vector-tensor multiplet of the dilaton).

3.2 Computing F_g : the One-Loop Integral

As in type IIA, the F_g of heterotic theory on $K3 \times T^2$ are couplings for the $R^2 F_+^{2g-2}$ terms, *i.e.* amplitudes involving two gravitons and $2g - 2$ graviphotons. They are computed using the odd spin structure on the worldsheet torus with insertion of vertex operators. The graviton vertices absorb the space-time fermions, while the graviphotons contribute $(p_R/\sqrt{2T_2U_2})^{2g-2}$, which will be summed with $q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2}$ over the $\Gamma_{2,2}$ lattice of the torus. The left-moving space-time (transverse) bosons and the extra free boson for the $U(1)$ current yield $1/\eta^3$, whereas the conformal blocks of the internal SCFT generate the $K3$ partition function $C_{K3} = \text{tr}_{RR} (-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}$. The latter is independent of \bar{q} , as usual due to Susy for the massive modes and to the absence of instanton contributions: these would be sensitive to deformations of hypermultiplet moduli (which contain the Kähler moduli of the compactification space), yet F_g is not sensitive to them!

¹²Except for F_0 and F_1 which also carry tree-level contributions

Overall [AGNT-95]:

$$F_g = \frac{1}{2\pi^2} \frac{1}{(g!)^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3} \frac{1}{\eta^3} \left\langle \prod_{i=1}^g \int_{T^2} d^2x_i Z^1 \partial Z^2(x_i) \prod_{j=1}^g \int_{T^2} d^2y_j \bar{Z}^2 \partial \bar{Z}^1(y_j) \right\rangle \\ \times C_{K3} \sum_{\Gamma_{2,2}} \left(p_R / \sqrt{2T_2 U_2} \right)^{2g-2} q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2}$$

where Z are the complex coordinates for the space-time right-moving bosons. The correlator for space-time bosons $\langle \int_{T^2} Z \partial Z \rangle$ can be summed over g with $(\frac{\lambda}{\tau_2})^{2g}$ and $\frac{1}{(g!)^2}$ to yield the function

$$\left(\frac{2\pi i \lambda \eta^3}{\vartheta_1(\tau, \lambda)} \right)^2 e^{-\frac{\pi \lambda^2}{\tau_2}}$$

which is modular invariant under $PSL(2, \mathbb{Z})$, *i.e.* under $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ and $\lambda \rightarrow \frac{\lambda}{c\tau+d}$. The full generating function for the amplitudes at all genera can then be written as

$$F(\lambda, T, U) = \sum_{g \geq 1} \lambda^{2g} F_g = \frac{1}{2\pi^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{C_{K3}}{\eta^3} \sum_{\Gamma_{2,2}} \left(\frac{2\pi i \lambda \eta^3}{\vartheta_1(\tau, \tilde{\lambda})} \right)^2 e^{-\frac{\pi \tilde{\lambda}^2}{\tau_2}} q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2}, \quad (3.1)$$

where $\tilde{\lambda} := \frac{p_R \tau_2}{\sqrt{2T_2 U_2}} \lambda$. In [MM-98], the quantity C_{K3}/η^3 takes the value of $E_4 E_6 / \eta^{24}$, in agreement with the $K3$ elliptic genus for a \mathbb{Z}_2 orbifold of heterotic compactification with gauge group $E_8 \times E_7 \times SU(2)$, as we shall see in (5.6) or (5.14). We note that the lattice sum is not restricted over $q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2}$, but also involves the p_R contained in $\vartheta_1(\tau, \tilde{\lambda})$.

Note also that *w.r.t.* $(\tau, \bar{\tau})$, τ_2 has weight $(-1, -1)$, $\frac{d^2\tau}{\tau_2}$ has weight $(-1, -1)$, $E_4 E_6 / \eta^{24}$ has weight $(-2, 0)$ while the remaining parts of the integrand have no well-defined weights. It is only upon expanding $\left(\frac{2\pi i \lambda \eta^3}{\vartheta_1(\tau, \tilde{\lambda})} \right)^2 e^{-\frac{\pi \tilde{\lambda}^2}{\tau_2}}$ in even powers of λ , say λ^{2k+2} , and keeping \bar{p}_R^{2k} for the lattice sum, that we obtain a suitable generalised theta function of weight $(1, 1 + 2k)$: $\Theta(T, U) := \sum_{\Gamma_{2,2}} \bar{p}_R^{2k} q^{\frac{1}{2}|p_L|^2} \bar{q}^{\frac{1}{2}|p_R|^2}$.

3.2.1 Lattice Reduction Technique

The evaluation of the above integral (3.1) over the fundamental domain is a *tour de force* [MM-98]. It is based on Borcherds' recursion formula for such automorphic integrals or *theta transforms*: at each step the lattice for the theta function is reduced by two dimensions. Although the results for F_g look rather messy, their holomorphic limit ($\bar{t} \rightarrow \infty$) is quite simple, and one does indeed recover the constant map contribution (1.15) which had also been obtained by computations on the type IIA side [BCOV-94]. The result for non-constant maps in the holomorphic limit $\bar{T}, \bar{U} \rightarrow \infty$ (T, U fixed) is similar to the \tilde{F}_g in (1.30):

$$\tilde{F}_g \sim \sum_{r > 0} c_{g-1}(r^2/2) Li_{3-2g}(e^{2\pi i r \cdot y})$$

where $y = (T, U)$ is the heterotic parameter, $r = (n, m)$ is a point in the lattice $\Gamma_{1,1}$, $r \cdot y = nU + mT$, $r^2 = 2nm$ and the condition $r > 0$ stands for $n, m \geq 0$ or $(n, m) = (1, -1)$ but not $(0, 0)$. The $c_{g-1}(n)$ are the Fourier coefficients of the modular function occurring in (3.1):

$$\frac{C_{K3}}{\eta^3} G = \sum_{\substack{g \geq 0 \\ m \geq -1}} c_{g-1}(m) \lambda^{2g-2} q^m,$$

$q := e^{2\pi i \tau}$. The computation suggests that the result should depend on the region of moduli space, and that a wall-crossing formula will relate different regions. Here, the wall happens to be the codimension one surface $T_2 = U_2$. By chance, the wall-crossing behaviour vanishes in the holomorphic limit, as expected from the type IIA side where this behaviour reflects the fact that two CY threefolds related by flop transition are birationally equivalent (*i.e.* same Hodge numbers).

Moreover, by organising the terms in the same way as the type IIA result, [MM-98] were able to extract from \tilde{F}_2 the number of genus 2 curves on a particular CY threefold. To this end, they first identified $-2 \sum_{r>0} c(r^2/2) \text{Li}_{-1}(e^{2\pi i r \cdot y})$ as the term corresponding to rational curves, with $c(n)$ the Fourier coefficients of $E_4 E_6 / \eta^{24}$ – these had already been seen in a previous heterotic approach [HM-95] (see below) as counting the number of rational curves. Since the $c(n)$ vanish for $n < -1$, the condition $r > 0$ above can be replaced by the positive root condition of the $\Gamma_{1,1}$ lattice for the monster Lie algebra, *i.e.* the condition $n > 0$; or $n = 0, m > 0$. The latter meaning was adopted in [HM-95].

3.2.2 Attempt Using Properties of Jacobi Forms

Integrals of elliptic genera over the fundamental domain have also been tackled from a different perspective in [N-98]. Rather than integrating the usual elliptic genus, the latter author slightly altered that index to obtain the so-called *new supersymmetric index*, and studied its integral. He showed that this new index, which is a trace over the left- and right-moving Ramond sectors, conveniently factorises when the compactification space is a product $T^2 \times K$ for some two-fold K :

$$\text{tr}_{R \times R} (-1)^F J_0 \bar{J}_0 q^{L_0 - \frac{\hat{c}}{8}} \bar{q}^{\bar{L}_0 - \frac{\hat{c}}{8}} = \left(\sum_{(p_L, p_R) \in \Gamma_{2,2}} q^{\frac{1}{2} p_L^2} \bar{q}^{\frac{1}{2} p_R^2} \right) \left(\text{tr}_K (-1)^F q^{L_0 - \frac{\hat{c}}{8}} \bar{q}^{\bar{L}_0 - \frac{\hat{c}}{8}} \right), \quad (3.2)$$

where \hat{c} is the dimension of the target space (3 in our case). The bracketed sum is the famous bosonic partition function for the torus, coming from the trace over the torus part of the threefold $K \times T^2$. The second part comes from the two-fold K and has to do with its elliptic genus at $z = 0$, $\Phi(\tau, 0)$.

The full elliptic genus reads $\Phi(\tau, z) = \text{tr}_{R \times R} (-1)^F q^{L_0 - \frac{\hat{c}}{8}} y^{J_0}$, see also (5.4), and is a weak Jacobi form of weight 0 and index $\hat{c}/2$ (see appendix A). For twofolds with SU_2 holonomy (two-tori, $K3$ surfaces), its expression is well-known [KKY-93]; for $\hat{c} = 1, 2, 3$ theories, $\Phi(\tau, 0)$ boils down to the *Witten index* of the theory, *i.e.* to $\text{tr} (-1)^F = \chi(K)$, the Euler character of the target space. For general models, one recovers the Euler character only upon taking the limit $\lim_{\tau \rightarrow i\infty} \Phi(\tau, 0)$.

However, in [N-98], $\Phi(\tau, 0)$ was not merely the Euler character, because a Wilson line was introduced. This inserted an extra compactification modulus (next to T and U for the torus) which prevented from writing the CY threefold as a direct product $K \times T^2$. Thus, the decomposition property (A.2) of Jacobi forms was used to compute the integral. The property states that a Jacobi form can be written as a finite linear combination of theta functions with modular forms as coefficients. For $z = 0$ the claim reduces to

$$\Phi(\tau, 0) = \sum_{\mu \pmod{2m}} h_\mu(\tau) \sum_{r \equiv \mu \pmod{2m}} q^{r^2/4m},$$

and $m = \frac{\hat{c}}{2} = 1$. The crux is that the extra summation of $q^{r^2/4m}$ over r could be joined with $q^{\frac{1}{2} p_L^2}$ in the above (3.2), resulting in an overall sum over a $(3, 2)$ lattice rather than just over $\Gamma_{2,2}$. In other words, one simply ended up with a new Siegel theta function of signature $(3, 2)$,

something one could integrate over the fundamental domain – using the well-known technique of ‘unfolding’.

Do we stand a chance for a similar trick for the integral in (3.1) ? Indeed, we recognise $-\eta^6/\vartheta_1(\tau, z)^2$ to be the inverse of the weak Jacobi form $A_{-2,1}$ studied in appendix A.4 of [G-04]. Unlike for the elliptic genus above, z is not set to 0 but contains factors of λ (string coupling) and τ_2 . But this should not discourage us, since these τ_2 factors may be joined with the τ_2 factors $q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2} = e^{\pi i \tau p^2 - 2\pi \tau_2 p_R^2}$ occurring in the Siegel theta function.

Yet a more serious point flaws our attempt to solve the integral by changing the Siegel theta function into a theta function of signature (3, 2), namely the fact that inverses of Jacobi forms have negative index and hence do not enjoy the periodicity property (A.1) nor the decomposition into linear combinations of theta functions (A.2). This too is explained in appendix A.4 of [G-04]. So the trick of [N-98] cannot be used here.

4 Automorphic Properties

This section deals with the mathematical aspects of the full GW potential in product form, and is essentially mathematical – though the first three sections present results without derivations. The evaluation of an integral over the fundamental plane [HM-95] yields the logarithm of a infinite product, of which Borcherds [B-95] had already predicted the automorphic properties (though obvious in this approach). The generalisation to arbitrary GW potentials is tried with Borcherds’ lifting of Jacobi forms to automorphic forms, but remains inconclusive. Prerequisites are section 1 and familiarity with infinite products à-la Borcherds; results will not be used in later sections.

4.1 Torus as Target Space

If we replace our three-dimensional Calabi-Yau space by a one-dimensional one, *i.e.* by a mundane elliptic curve, then the Gromov-Witten problem boils down to the Hurwitz problem of counting covers of a Riemann surface. The *free energies* F_g have been similarly defined, and an explicit expression for the *partition function* $Z := \exp \sum_1 \lambda^{2g-2} F_g$ was given in [Dou-93]. It involves a generalised theta function, thereby ensuring modular properties of the F_g ’s. For example, $F_1 = -\log \eta(q)$, and F_2 is a linear combination of Eisenstein series [Ru-94]. This is to be compared with the GW potential $F_1 = -\sum_d n_d^1 \log \eta(q^d)$ where we have an additional sum over the homology class of the image curve. Of course, in the case of covers of an elliptic curve, there is no degree to keep track of, and t is the Kähler modulus of the flat torus. Mirror symmetry relates t to the complex modulus τ of the mirror elliptic curve, which accounts for its modular covariance under $PSL(2, \mathbb{Z})$. More generally, F_g is a quasi-modular form of weight $6g - 6$. One might then wonder whether our present GW potentials enjoy similar modular properties.

However, the occurrence of the \tilde{n}_d^r in (1.30) and the fact that d is a tuple (and not just an integer) spoil the following naive hope of modular properties:

$$F_g \sim \sum_{d>0} \text{Li}_{3-2g}(q^d) = \sum_n n^{2g-3} \frac{q^n}{1-q^n} = E_{2g-2}$$

yielding the Eisenstein series of weight $2g - 2$. So one wonders if a favourable choice for the integers \tilde{n}_d^r and for the summation over d would keep the modular properties. This was indeed the case for part of the results of [HM-95], to which we now turn.

4.2 Harvey-Moore and the Theta Transforms

In that work, the prepotential for heterotic string theory, which coincides with F_0 , was computed by hand. By equating the Wilsonian coupling with the one-loop coupling renormalisation (5.2),

one obtains a differential equation for the prepotential, involving a class of integrals over the fundamental domain. The integrand can be replaced by the “new susy index”, as in (5.3), which can be explicitly computed and yields a generalised theta function (or lattice function $\Gamma_{n+2,2}$) times a modular function, see for instance (5.17) or (5.19). Thus these are integrals of the form of a *theta transform*, i.e. of an integral over the fundamental domain of a Siegel theta function times an almost holomorphic modular function of weight $-s/2$ with Fourier expansion $F(q) = \sum c(n, k) q^n \tau_2^{-k}$, $q = e^{2\pi i \tau}$, with summation running over $n \geq -n_0$ and $k = 0, 1, \dots, k_0$ for some non-negative integers n_0 and k_0 :

$$\Phi_{s+2,2}(y) := \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \Theta(\tau, y) F(\tau) .$$

This is roughly the Howe correspondence, sending automorphic functions F for $SL(2, \mathbb{Z})$ to automorphic functions Φ for $O_{s+2,2}(\mathbb{Z})$. In general, F is allowed to be modular covariant at level N (or, equivalently, vector-valued at level 1) and up to a character, to have poles at cusps and even to have rational powers of q in its expansion. For instance, the function $1/\tau_2$ is modular of weight $(1, 1)$. The theta function of weight $(s/2 + 1, 1)$ is defined for an even self-dual lattice $\Gamma_{s+2,2}$, (with $8|s$):

$$\Theta(\tau, y) = \sum_{p \in \Gamma_{s+2,2}} q^{p_L^2/2} \bar{q}^{p_R^2/2}$$

where y lies in the Grassmanian $G(s+2, 2) = O(s+2, 2)/O(s+2) \otimes O(2)$ sometimes also called the generalised upper half plane $\mathcal{H}^{s+1,1} \cong \mathbb{R}^{s+1,1} + i C_+^{s+1,1}$ for the positive light cone $C_+^{s+1,1}$. Then y corresponds to a choice of a positive definite $s+2$ dimensional subspace of $\mathbb{R}^{s+2,2} \cong \Gamma_{s+2,2} \otimes \mathbb{R}$, so that every lattice vector p can be projected onto left and right (or positive and negative definite) subspaces: $p = (p_L, p_R)$ with $p^2 = p_L^2 - p_R^2$ and $p_{L,R}^2 \geq 0$. Note that even though $y \in \mathcal{H}$, the function Θ – and hence Φ – is actually automorphic under $\text{Aut}(\Gamma_{s+2,2}) = O_{s+2,2}(\mathbb{Z})$. In a more general treatment [B-96], Θ also depends on a homogeneous polynomial and its defining lattice need not be self-dual (in which case the modular properties are recovered by considering vector-valued theta functions).

4.3 The Result for the Prepotential

In the easier case where $F(q) =: \sum_{n \geq -n_0} c(n) q^n$ is a meromorphic modular form, [HM-95] have computed the integral by a generalisation of the technique of [DKL-91] via unfolding the fundamental domain. It yields the logarithm of an automorphic product à la Borchers for the lattice $\Gamma_{s+1,1}$:

$$\Phi_{s+2,2}(y) = -2 \log \left| e^{-2\pi \rho \cdot y} \prod_{r > 0} (1 - e^{-2\pi r \cdot y})^{c(-r^2/2)} \right|^2 + c(0) (-\log[-(\Re y)^2] - \mathcal{K}),$$

with r the positive roots of the lattice $\Gamma_{s+1,1}$, ρ the so-called Weyl vector of the lattice, and \mathcal{K} some insignificant constant.

According to Borchers¹³ [B-95], if $F(q)$ has weight $-s/2$ and the $c(n)$ are integers (with $24|c(0)$ if $s=0$), then such a product can be analytically continued to a meromorphic automorphic form of weight $c(0)/2$ on $O_{s+2,2}(\mathbb{Z})$; its zeroes and poles lie on rational quadratic divisors $ay^2 + r \cdot y + c = 0$ ($a, c \in \mathbb{Z}$). Note that inside the radius of convergence, there are no poles and all zeroes lie on linear divisors $r \cdot y + c = 0$ ($c \in \mathbb{Z}$).

A second type of integral, $\tilde{\Phi}_{s+2,2}(y)$, in which $F(q)$ is replaced by $F(q)(E_2(q) - \frac{3}{\pi\tau})$, was also computed in [HM-95] and yielded a similar result containing the above $\log \prod_{r > 0} (1 -$

¹³Note our exponent of $-2\pi r \cdot y$ as opposed to Borchers' $2\pi i r \cdot y$.

$e^{-2\pi r \cdot y} c(-r^2/2) = \sum_{r>0} c(-\frac{r^2}{2}) \log(1 - e^{-2\pi r \cdot y}) = - \sum_{r>0} c(-\frac{r^2}{2}) \text{Li}_1(e^{-2\pi r \cdot y})$ together with further polylogs Li_2 and Li_3 .

Hence for the linear differential equation where these two integrals occur, we also expect its solution to share the automorphic properties. This solution is the one-loop prepotential; explicitly:

$$F_0(y) := h^{(1)}(y) = \frac{1}{384\pi^2} \tilde{d}_{ijk} y^i y^j y^k - \frac{1}{2(2\pi)^4} c(0) \zeta(3) - \frac{1}{(2\pi)^4} \sum_{r>0} c(-\frac{r^2}{2}) \text{Li}_3(e^{-2\pi r \cdot y}) \quad (4.1)$$

Here $y = (\vec{y}, T, U) \in \mathcal{H}^{s+1,1}$ and $y^2 = \vec{y}^2 + 2TU$. The symmetric tensor \tilde{d}_{ijk} depends on the Weyl vector and the particular algebra at hand; it is irrelevant for us. This prepotential was obtained in the context of heterotic compactifications on $K3 \times T^2$, with standard embedding yielding a gauge group $E_7 \times SU(2) \times E_8 \times U(1)^4$. The E_8 part can be broken by the introduction of s Wilson lines (same s as in $\mathcal{H}^{s+1,1}$). [HM-95] considered the cases $s = 0$ and 8 .

For the case $s=0$, $y = (T, U)$, $c(n)$ are the Fourier coefficients of $F_{s=0} := E_4 E_6 / \eta^{24}$ (which is $\Gamma_8 = E_4$ times $F_{s=8}$) and the sum runs over all positive roots $r = (n, m)$ of the monster Lie algebra: $m > 0$; or $m = 0, n > 0$. The Yukawa coupling $\partial_U^3 h^{(1)}$ agrees with another expression [AFGNT-95]:

$$\partial_U^3 F_0(T, U) = -\frac{1}{2\pi} \left(1 - \sum_{r>0} c(kl) l^3 \frac{e^{-2\pi(kT+lU)}}{1 - e^{-2\pi(kT+lU)}} \right) = -\frac{1}{2\pi} \frac{E_4(iU)E_4(iT)E_6(iT)}{(J(iT) - J(iU)\eta(iT)^{24})}$$

This involves only modular forms in T, U separately and is of weight $(-2, 4)$ in (T, U) . Note that the $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ symmetry is isomorphic to the symmetry group $SO(2, 2, \mathbb{Z})/\mathbb{Z}_2$, where \mathbb{Z}_2 stands for the exchange of T and U , and $SO(2, 2, \mathbb{Z})$ is the automorphic group of $\Gamma_{2,2}$. How much of these modular properties are then left for the prepotential $h^{(1)}$ itself?

For the case $s=8$, to which we turn for the remainder of this section, $c(n)$ are the Fourier coefficients of $F_{s=8} := E_6 / \eta^{24}$ – see eqn (5.14) – and the sum runs over the positive roots $r = (\vec{r}, n, m)$ of the E_{10} Lie algebra where \vec{r} is itself a positive element of Γ_8 (the root lattice of E_8).

In the particular \mathbb{Z}_2 orbifold limit of the $K3$, the massless spectrum consists of **(56, 2, 1)** + 8 **(56, 1, 1)** + 32 **(1, 2, 1)** + 4 **(1, 1, 1)** in the gauge group $E_7 \times SU(2) \times E_8 \times U(1)^4$, as argued in the paragraph containing (5.15). These total of 628 hypermultiplets and 388 vector multiplets ($388 = \text{rank of the gauge group}$) are too high to hope for an easily describable CY dual on which to compactify the type II A theory. For the latter, we need few Kähler moduli, *i.e.* few vector multiplets on the heterotic side, *i.e.* a smaller gauge group. As announced, we assume that E_8 is broken to $U(1)^8$ by introducing $s = 8$ Wilson loops; we shall further Higgs completely $E_7 \times SU(2)$ to end up with the gauge group $U(1)^{12}$. The Higgsing will cost us $133 + 3$ scalars to give mass to the gauge fields in the adjoint (as outlined in the examples of section 7 of [G-04]). Thus we are left with 492 hypers and 12 vectors, which begs for a CY threefold with $h^{21} = 491$ and $h^{11} = 11$.

Such a CY X fortunately exists; it is a degree 84 hypersurface in $\mathbb{P}(1, 1, 12, 28, 42)$, *i.e.* a $K3$ fibration over \mathbb{P}^1 where the $K3$ is a degree 42 hypersurface in $\mathbb{P}(1, 6, 14, 21)$. In this model for topological string theory, we see that the above prepotential F_0 has remarkably the shape desired for counting instantons – recall the Li_3 from (1.28). Then we see that the rational holomorphic curves in the fibres of the CY are parametrised by the positive roots r of the E_{10} root lattice, and their number is given by $c(-r^2/2)$. So the only question left is whether F_0 (or any derivative thereof) in this case enjoys similar modular properties as in the case $s = 0$.

Furthermore, one is driven to ponder on the following issues: What is the relation between the sum over $d > 0$ in (1.28) and the sum over $r > 0$ in (4.1)? Would the F_g ($g > 0$) in (1.30) enjoy modular or automorphic properties? Could they also be expressed as a sum over

$\Gamma_{s+2,2}$ or over a root lattice of some algebra, rather than over $H_2(X, \mathbb{Z}) = \mathbb{Z}^{h^{11}}$? If so, what is the relation between the CY X and the lattice of which the positive roots govern the counting of holomorphic curves? Would the \tilde{n}_d^r be the (integer) coefficients of some nearly-holomorphic modular forms, like in (4.1)?

We conjecture that this should indeed be the case. The work of [KY-00] sheds some light in this direction.

4.4 Extension of the Moduli Space

The way to convert the general form of (1.30) into a modular product à la Borchers is to generalise Borchers' lifting of a Jacobi form $\Phi_0(\tau, z)$ of weight zero for a positive definite lattice to a lifting of a form $\Phi_0(\tau, z, \lambda)$ defined on a *Lorentzian* lattice Γ . This extension comes along with the extension of the moduli space to include the string coupling constant λ next to the moduli t_i : $H^2(X, \mathbb{Z}) \oplus H^0(X, \mathbb{Z})$. The crucial idea in [KY-00] is to rewrite the GW potential as a sum over Hecke operators V_l acting on Φ_0 :

$$F = \sum_{g \geq 0} \lambda^{2g-2} F_g(q_i) = \sum_{g \geq 0} \lambda^{2g-2} \mathcal{F}_g(p, \tau, z) = \sum_{l \geq 0} p^l \Phi_0|_{V_l}(\tau, z, \lambda),$$

i.e. to absorb the parameter λ into the lattice and to express F_g using variables (p, τ, z) instead of the tuple $q_i = e^{t_i}$. The new lattice is defined to be $\Gamma := Q^\vee(m) \oplus \langle -2 \rangle$ where Q^\vee is the coroot lattice of a simple Lie algebra \mathfrak{g} and $Q^\vee(m) = (Q^\vee, m\langle \cdot, \cdot \rangle)$. The lattice Γ^* contains points (γ, j) which will be multiplied with the variables $(z, \lambda) \in \Gamma_{\mathbb{C}}$ to yield $e^{2\pi i(\gamma \cdot z + j\lambda)} =: \zeta^\gamma y^j$ in the Fourier series to follow. Let also $q := e^{2\pi i\tau}$.

To further understand the procedure, we give here concrete formulae: Let $\phi_{-2,m}$ be a nearly-holomorphic Jacobi form of weight -2 and index m , invariant under the action of the Weyl group of \mathfrak{g} . The function $(\vartheta_1(\tau, \lambda)/\eta(\tau)^3)^2$ is itself a weak Jacobi form of weight -2 and index 1. We define Φ_0 and its Fourier and Taylor coefficients as follows:

$$\begin{aligned} \Phi_0(\tau, z, \lambda) &:= -\phi_{-2,m}(\tau, z) \frac{\eta(\tau)^6}{\vartheta_1(\tau, \lambda)^2} \\ &=: \sum_{\substack{n \geq -n_0 \\ (\gamma, j) \in \Gamma^*}} D(n, \gamma, j) q^n \zeta^\gamma y^j \\ &=: -\sum_{g=0}^{\infty} \lambda^{2g-2} \varphi_{2g-2,m}(\tau, z) \end{aligned}$$

The $\varphi_{2g-2,m}(\tau, z)$ are quasi Jacobi forms of weight $2g - 2$ since the λ -expansion of $1/\vartheta_1^2$ has (quasi) modular forms as coefficients.

Then one can check that

$$\Phi_0|_{V_l}(\tau, z, \lambda) = -\sum_{g \geq 0} \lambda^{2g-2} \varphi_{2g-2,m}|_{V_l}(\tau, z), \quad (\forall l \geq 0)$$

which allows us to express \mathcal{F}_g directly in terms of the $\varphi_{2g-2,m}$:

$$\begin{aligned} \mathcal{F}_g(p, \tau, z) &= \sum_{l \geq 0} p^l \varphi_{2g-2,m}|_{V_l}(\tau, z) \\ &= \dots \\ &= \frac{c_g(0, 0)}{2} \zeta(3 - 2g) + \sum_{(l, n, \gamma) > 0} c_g(l, n, \gamma) \text{Li}_{3-2g}(p^l q^n \zeta^\gamma), \end{aligned} \tag{4.2}$$

where $(l, n, \gamma) > 0$ means $l > 0$ or $l = 0, n > 0$ or $l = n = 0, \zeta > 0$, and the c_g are the Fourier coefficients of $\varphi_{2g-2,m}$:

$$\varphi_{2g-2,m}(\tau, z) =: \sum_{n,\gamma} c_g(n, \gamma) q^n \zeta^\gamma.$$

A similar action of the Hecke operators is valid on any Jacobi form Φ_k of weight k :

$$\sum_{l=0}^{\infty} p^l \Phi_k|_{V_l}(\tau, z, \lambda) = \sum_{(l,n,\gamma,j)>0} D(ln, \gamma, j) \text{Li}_{1-k}(p^l q^n \zeta^\gamma y^j),$$

modulo constant terms. The benefit of having chosen Φ_0 of weight 0 is that the Li_1 is but a logarithm, and the *partition function* can thus be expressed as an infinite product:

$$\mathcal{Z}(\sigma, \tau, z, \lambda) = e^{\mathcal{F}} = \exp \sum_{g \geq 0} \lambda^{2g-2} \mathcal{F}_g \sim \prod_{\substack{(l,n,\gamma,j)>0 \\ (\gamma,j) \in \Gamma^*}} (1 - p^l q^n \zeta^\gamma y^j)^{D(ln,\gamma,j)}$$

modulo the Weyl vector. Recall that $p = e^{2\pi i \sigma}$, $q = e^{2\pi i \tau}$, $\zeta = e^{2\pi i z}$, $y = e^{2\pi i \lambda}$.

4.5 Mapping the Moduli q_i to (u, p, q, ζ)

The question arises as to how do we map the tuple $q_i = e^{t_i}$ to the variables (p, τ, z) . We now carry this out for the special case of a CY threefold X that can be realised both as $K3$ fibration over \mathbb{P}^1 and as an elliptic fibration over a surface W^2 . We choose our simple Lie algebra \mathfrak{g} with coroot lattice Q^\vee to be of rank $s = h^{1,1}(X) - 3$ and such that the Piccard lattice of a general $K3$ fibre is isomorphic to $\Gamma_{1,1} \oplus Q^\vee(-m)$ for the even self-dual Lorentzian lattice $\Gamma_{1,1}$ of signature $(1, 1)$. In other words, we need three more variables next to the lattice variables ζ^γ to correspond to the Kähler moduli $t_1, \dots, t_{h^{1,1}}$ of X . Let those three be u, p, q , where $u := e^{t_1}$ is new here: it is related to $q_1 = e^{t_1}$ and will become redundant in the limit of large base \mathbb{P}^1 (*i.e.* $t_1 \rightarrow \infty$). This is why u does not appear in the argument of $\mathcal{F}_g = \mathcal{F}_g(p, \tau, z)$. We furthermore assume that $\eta(\tau)^{24} \phi_{-2,m}(\tau, z)$ is a Jacobi function of weight 10 and index m , equal to $-2E_4 E_6$ at $z = 0$, and such that $c_0(0, 0) = -\chi(X)$.

The precise mapping between the t_i 's and (u, p, q, ζ) was suggested by [KY-00] as

$$\begin{aligned} t_1 &= \log u - \log q, \\ t_2 &= \log p - \log q, \\ t_3 &= \log q - (\gamma_0, \log \zeta), \\ t_{i+3} &= (\Lambda_i, \log \zeta), \quad (i = 1, \dots, s), \end{aligned}$$

for γ_0 some positive weight and Λ_i ($i = 1, \dots, s$) the fundamental weights of \mathfrak{g} . Thus

$$\begin{aligned} q^t &= e^{t \cdot d} = e^{t_1 d_1} e^{t_2 d_2} e^{t_3 d_3} e^{\sum_1^s t_{3+i} d_{3+i}} \\ &= (u^{d_1} q^{-d_1}) (p^{d_2} q^{-d_2}) (q^{d_3} \zeta^{-\gamma_0 d_3}) (\zeta^{\sum_1^s \Lambda_i d_{3+i}}) \\ &= u^{d_1} p^{d_2} q^{-d_1 - d_2 + d_3} \zeta^{\gamma_0 d_3 + \sum \Lambda_i d_{3+i}}, \end{aligned}$$

and as $u \rightarrow 0$, only $d_1 = 0$ contributes, and thus the identification is:

$$p^{d_2} q^{d_3 - d_2} \zeta^{\gamma_0 d_3 + \sum \Lambda_i d_{3+i}} \cong p^l q^n \zeta^\gamma.$$

When summing over $d > 0$, *i.e.* over all d_2, \dots, d_l not all zero, we see that the following three cases occur:

$$\begin{aligned} l &> 0, \text{ and } n, \gamma \text{ arbitrary,} \\ l &= 0, n > 0, \gamma \text{ arbitrary,} \\ l &= n = 0, \gamma > 0. \end{aligned}$$

These are just the conditions $(l, n, \gamma) > 0$ that we needed, as in (4.2). In other words, we have attained our goal of rewriting F_g in (1.30) as a sum over positive roots of a lattice $\Gamma_{1,1} \oplus (Q^\vee)^*(\frac{1}{m})$. At least, this is what [KY-00] conjectured:

$$F_g = \mathcal{F}_g \quad (g \geq 2).$$

(plus extra constant terms at $g = 0, 1$: F_0^{const}, F_1^{const}). This translates into their main conjecture for the partition function:

$$Z(q_i, \lambda) = \mathcal{Z}(\sigma, \tau, z, \lambda) = \exp(x^{-2} F_0^{const} + F_1^{const}) \left[\prod_{(l, n, \gamma, j) > 0} (1 - p^l q^n \zeta^\gamma y^j)^{D(l, n, \gamma, j)} \right],$$

with $(l, n) \in \Gamma_{1,1}$, $(\gamma, j) \in \Gamma^*$. This is a modular product à la Borcherds for the lattice $\Gamma_{1,1} \oplus \Gamma^*$, with the only difference that Γ here is a Lorentzian lattice instead of a positive definite one. The term in the exponential is then the corresponding Weyl vector. The vector $v = (\sigma, \tau, z, \lambda)$, such that $e^{2\pi i v} := pq\zeta y$, would then be element of the Grassmanian corresponding to the modular product. Would this be $O(s+2, 3)$?

5 Threshold Corrections for Heterotic Orbifolds

This crowning section is rather intricate, both mathematically and physically. It aims at bringing together the different objects touched upon in the two previous sections, as well as giving the details for some computations that we left out. Thus, we shall come back on the “new susy index” (3.2), explain our explicit values for the $K3$ elliptic genus C_{K3}/η^3 in (3.1) and for the functions $F(q) = \sum c(n)q^n$ in section 4.3. In fact, the latter are nothing but results that pop up in computations of *threshold corrections*.

These are upper-half-plane integrals and are presented in (5.2) of the first subsection, including a gauge theory factor which we shall neglect until the very last subsection.

An explicit expression, $\Gamma_{10,2} E_6 / \eta^{24}$, for the integrand of the threshold correction (5.2) is given in (5.14), after a direct but cumbersome calculation via the orbifold partition functions of section 5.3 for $K3 \times T^2$ (we assume familiarity with partition functions for bosons/fermions on several compactification spaces). The same expression can be obtained after first re-writing the original integrand (5.2) as the “new susy index” (5.3) and then evaluating the latter in the particular case of $K3 \times T^2$ to obtain (5.6).

The last subsection 5.4 finally incorporates the gauge theory factor of the integrand of the threshold correction. The results there are just the functions $F(q)$ or $F(q)\hat{E}_2$ used in the integrands of section 4.3 that yielded the infinite products à-la Borcherds. Similarly, the $K3$ elliptic genus C_{K3}/η^3 of (3.1) will be seen here to yield $E_4 E_6 / \eta^{24}$ as in (5.14). Thus this chapter connects a good part of what we introduced previously.

5.1 Effective Field Theory and One-Loop Threshold Corrections

Threshold corrections to coupling constants are the differences between calculations in two separate frameworks: the fundamental theory (or field theory, FT) and the effective field theory (EFT). The latter takes only light states into account (the massless NS-NS fields $G_{\mu\nu}, B_{\mu\nu}, \phi$) and computes correctly in the low-energy range, *i.e.* up to energies neighbouring the masses of the heavy states (which occur at the scale of the string length). At tree-level, it computes the *on-shell* scattering amplitudes for light states up to terms that vanish by the equations of motion, *i.e.* it integrates out the heavy states. The FT on the other hand takes both light and heavy states into account; so the difference between the FT and EFT result is precisely the

contribution from heavy states that the EFT misses out. This contribution should be added to the loop-expansion in the EFT and is called the *threshold correction* Δ_i . The subscript i refers to the gauge group for which we compute the gauge coupling.

At tree level, the gauge coupling constant is equal to the string coupling. Denoting the one-loop correction by Δ_i , we have to one-loop order:

$$\frac{1}{g_i^2} = \frac{k_i}{g_{\text{str}}^2} + \Delta_i, \quad (5.1)$$

The constant k_i is the central element for the left-moving algebra which generates the gauge group G_i . Since G_i is non-abelian, we set $k_i = 1$. The threshold correction itself can be computed for an N=2 heterotic compactification on $K3 \times T^2$ to be (see [G-04] for a sketch)

$$\Delta_i = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\frac{-i}{\pi|\eta|^4} \sum_{\text{even}} (-1)^{a+b} \bar{\partial}_{\bar{\tau}} \left(\frac{\bar{\vartheta}^{[a]}_{[b]}}{\bar{\eta}} \right) \text{Tr}_{\text{int}} \left[Q_i^2 - \frac{k_i}{4\pi\tau_2} \right] [^a_b] - b_i \right). \quad (5.2)$$

The subscript i refers to one of the several factors of the gauge group, while “even” spin structures means $(a, b) \neq (1, 1)$ ($a, b = 0, 1$). The constant b_i is called the *beta function coefficient* and represents the constant term of the q -expansion of the integrand (see also appendix E of [G-04]); subtracting it renders the integral IR finite. Note that we have suppressed for convenience the factors of q in the trace: the proper expression should contain also $C_{\text{int}}[^a_b] := \text{tr} (-1)^{b\bar{J}_0} q^{L_0 - \frac{11}{12}} \bar{q}^{\bar{L}_0 - \frac{3}{8}} [^a_b]$ for the $(c, \bar{c}) = (22, 9)$ internal theory, where \bar{J}_0 coincides with the right-moving fermion number F_R . The notation $[^a_b]$ stands for the spin structure, see also (5.11).

5.2 Threshold via New Susy Index and $K3$ Elliptic Genus

An alternative way of writing the integrand (5.2) is as a “new susy index” (3.2) (see again [G-04] for a sketch)

$$\frac{16\pi^2}{g_i^2} \Big|_{1\text{-loop}} = \frac{-1}{\eta^2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\text{Tr}_{R, \text{int}} \left(\bar{F}(-1)^{\bar{F}} q^{L_0 - \frac{11}{12}} \bar{q}^{\bar{L}_0 - \frac{3}{8}} \left[Q_i^2 - \frac{k_i}{4\pi\tau_2} \right] \right) - b_i \right), \quad (5.3)$$

with the usual group theory factor in square brackets. Indeed, this was the starting point in [HM-95].

The computation of this new susy index is quite different for the left- and right-moving sectors. In the right-moving sector, the algebra factorises into the direct sum of a $\bar{c} = 3$ N=2 SCA and a $\bar{c} = 6$ N=4 SCA, which simplifies considerably the computation of the new susy index: the result is a mere constant, $-2i$, thanks to the equal but opposite contributions of vector- and hypermultiplets towards $\bar{J}_0(-1)^{\bar{J}_0}$ (see [HM-95] or section 10.5 of [G-04]). In the left-moving sector, the new susy index enters in the form of the above elliptic genus $\text{tr}_R (-1)^{J_0 + \bar{J}_0} q^\Delta$, to which we now turn.

$K3$ Elliptic Genera

The elliptic genus of a $(c, \bar{c}) = (6, 6)$ heterotic sigma model on $K3 \times T^2$ is, geometrically, a double sum $\Phi(\tau, z) = \sum_{n,r} c_{n,r} q^n y^r$ whose coefficients are the indices of Dirac operators for certain vector bundles over $K3$. The elliptic genus also has a topological expression, given by a trace over the left and right Ramond sectors with $(-1)^F$ insertion:

$$\begin{aligned} \Phi(\tau, z) &:= \text{tr}_{R, R} (-1)^{F_L + F_R} q^{L_0 - 1/4} \bar{q}^{\bar{L}_0 - \bar{1}/4} y^{J_0} \\ &= 24 \left(\frac{\vartheta_3(z)}{\vartheta_3} \right)^2 + 2 \frac{\vartheta_2^4 - \vartheta_4^4}{\eta^4} \left(\frac{\vartheta_1(z)}{\eta} \right)^2, \end{aligned} \quad (5.4)$$

where the second expression was proved in [EOTY-89], [G-03], and corresponds to the unique weak Jacobi form of weight 0 and index 1 (see appendix A) with $\Phi(\tau, 0) = \chi(K3) = 24$. At the special values of $z = \frac{1+\tau}{2}, \frac{\tau}{2}, \frac{1}{2}$ and 0, we obtain specific topological invariants [EOTY-89], using (B.3),(B.4) and dropping the extra $q^{-1/4}$ and $-q^{-1/4}$ in the first two cases:

$$\begin{aligned}
\text{Dirac index :} \quad & \Phi_{\hat{A}}^+ := \text{tr}_{\text{NS,R}}(-1)^{F_R} q^{L_0-1/4} = 2\vartheta_3^2(\vartheta_2^4 - \vartheta_4^4)/\eta^6 \\
& \Phi_{\hat{A}}^- := \text{tr}_{\text{NS,R}}(-1)^{F_L+F_R} q^{L_0-1/4} = -2\vartheta_4^2(\vartheta_2^4 + \vartheta_3^4)/\eta^6 \\
\text{Hirzebruch genus :} \quad & \Phi_\sigma := \text{tr}_{\text{R,R}}(-1)^{F_R} q^{L_0-1/4} = 2\vartheta_2^2(\vartheta_4^4 + \vartheta_3^4)/\eta^6 \\
\text{Euler character :} \quad & \Phi_\chi := \text{tr}_{\text{R,R}}(-1)^{F_L+F_R} q^{L_0-1/4} = 24
\end{aligned} \tag{5.5}$$

Whence a shift $z \rightarrow z + \frac{\tau}{2}$ generates spectral flow $\text{R} \rightarrow \text{NS}$, while $z \rightarrow z + \frac{1}{2}$ is responsible for an additional factor of $(-1)^{F_L}$. The elliptic genus evaluated at specific points thus yields the partition function for different spin structures (boundary conditions in the left-moving sector); at $z = 0$, we obtain the Witten index – or the bosonic partition function if we have no spin structures.

“New Susy Index”

It remains us to compute explicitly the new susy index for the left-moving sector, that is to evaluate the above elliptic genus or to find the proper partition functions.

In our compactification on $T^2 \times K3$, we shall use bosonic formulation for one of the E_8 gauge groups and fermionic formulation (with sixteen left-moving fermions) for the other. The gauge bundle is a direct sum of a bundle on T^2 and one on $K3$, with flat connection or a.s.d. connection respectively. We choose to couple 12 fermions with the former connection and 4 with the latter, so as to obtain the desired $(c, \bar{c}) = (6, 6)$ heterotic sigma model on $K3$.¹⁴ The partition functions for the former fermions on T^2 are the familiar ϑ_i/η , $i = 1, 2, 3, 4$ for the NS/R sectors with/without $(-1)^F$ insertion. The bosonic realisation of the other E_8 factor yields the familiar theta function Γ_8/η^8 , which we join with the $\Gamma_{2,2}$ lattice of T^2 to obtain $\Gamma_{10,2}/\eta^{12}$. Taking into account the $-2i$ from the right-moving sector, and summing over all worldsheet boundary conditions of the left-moving sector, we obtain for the whole “new susy index” of the internal theory:

$$\begin{aligned}
& \text{tr}_{\text{R}} \bar{J}_0 e^{i\pi(J_0 - \bar{J}_0)} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \\
& = -2i \frac{\Gamma_{10,2}}{\eta^{12}} \left(\left(\frac{\vartheta_3}{\eta} \right)^6 \Phi_{\hat{A}}^+ + \left(\frac{\vartheta_4}{\eta} \right)^6 \Phi_{\hat{A}}^- + \left(\frac{\vartheta_2}{\eta} \right)^6 \Phi_\sigma + \left(\frac{\vartheta_1}{\eta} \right)^6 \Phi_\chi \right) \\
& = -2i \frac{\Gamma_{10,2}}{\eta^{12}} \frac{2}{\eta^{12}} \left(\vartheta_3^8(\vartheta_2^4 - \vartheta_4^4) - \vartheta_4^8(\vartheta_2^4 + \vartheta_3^4) + \vartheta_2^8(\vartheta_3^4 + \vartheta_4^4) \right),
\end{aligned}$$

hence

$$\text{tr}_{\text{R}} \bar{J}_0 e^{i\pi(J_0 - \bar{J}_0)} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} = 8i \Gamma_{10,2} \frac{E_6}{\Delta}, \tag{5.6}$$

where we noted that the last bracket is but $\sum_{i \neq j} \vartheta_i^8 \vartheta_j^4$ with a minus sign if $2i + j > 8$, which is just $-2E_6$ by (B.10).

5.3 Threshold via Heterotic Orbifold Partition Function

We will now derive the same result for the “new susy index”, but this time starting from the expression (5.2) for the partition function

$$Z_{D=4}^{\text{het}} = \frac{1}{\tau_2 \eta^2 \bar{\eta}^2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \frac{\bar{\vartheta}_b^{[a]}}{\bar{\eta}} C_{\text{int}[b]}^{[a]} \tag{5.7}$$

¹⁴In general, $(16 - 2n)$ fermions on T^2 plus $2n$ on $K3$ gives a $(c, \bar{c}) = (4 + n, 6)$ model. Hence each fermion coupled to $K3$ increases the left-moving central charge by $1/2$. We have $n = 2$.

with the appropriate internal contribution $C_{\text{int}}[b]^a = \text{tr}_{\text{int}} (-1)^{b\bar{J}_0} q^\Delta \bar{q}^{\bar{\Delta}} [b]^a$, but still without the gauge theory factor with the Casimir operator. It is encouraging to see the alternative result (5.14) agree with (5.6).

The internal contribution consists of the partition functions for the T^2 bosons ($\Gamma_{2,2}(T, U)/\eta^2 \bar{\eta}^2$) and fermions ($\frac{1}{2} \sum_{a,b} (-1)^{a+b+ab} \bar{\vartheta}[b]^a / \bar{\eta}$), for the $K3$ bosons and fermions, as well as for the gauge bundle and its two E_8 factors which we take in the bosonic realisation ($\Gamma_8/\eta^8 = E_4/\eta^8$) and fermionic realisation ($\frac{1}{2} \sum_{\gamma,\delta} \vartheta^8[\gamma]/\eta^8$) respectively. Since the partition function (or the elliptic genus) is a topological object, it does not depend on the hypermultiplet moduli, so we will choose a limit for these moduli where the $K3$ surface is described by a \mathbb{Z}_2 orbifold breaking the gauge group to $E_8 \times E_7 \times SU(2)$. This has a well-known partition function for the bosonic (4,4) blocks:

$$Z_{(4,4)}[0] = Z(R) = \frac{\Gamma_{4,4}(G, B)}{\eta^4 \bar{\eta}^4}, \quad Z_{(4,4)}[g] = 2^4 \frac{\eta^2 \bar{\eta}^2}{\vartheta^2[1-h] \bar{\vartheta}^2[1-h]} \quad (h, g) \neq (0, 0), \quad (5.8)$$

where the lattice function depends on the metric G_{ij} and the B-field B_{ij} :

$$\begin{aligned} \frac{\Gamma_{4,4}(G, B)}{\eta^4 \bar{\eta}^4} &:= \sum_{m,n \in \mathbb{Z}^4} \frac{q^{p_L^2/2} \bar{q}^{p_R^2/2}}{\eta^4 \bar{\eta}^4}, \quad p_{L,R}^i := \frac{G^{ij}}{\sqrt{2}} (m_j + (B_{jk} \pm G_{jk}) n_k) \\ &= \frac{\sqrt{\det G}}{(\sqrt{\tau_2} \eta \bar{\eta})^4} \sum_{m,n \in \mathbb{Z}^4} \exp \left(-\frac{\pi}{\tau_2} (G_{ij} + B_{ij}) (m_i + n_i \tau) (m_j + n_j \bar{\tau}) \right) \end{aligned} \quad (5.9)$$

with $p_{L,R}^2$ the inner product *w.r.t.* the metric, *i.e.* $p_{L,R}^i G_{ij} p_{L,R}^j$. (Note that our metric G_{ij} has absorbed a factor of R_i (orbifold radii) compared to other conventions.)

Similarly, the compact $K3$ fermions ($\bar{\vartheta}^2/\eta^2$) are twisted in the (4,4) blocks: $\bar{\vartheta}[b+h] \bar{\vartheta}[b-h]/\eta^2$. Additionally, the orbifold projection on the fermionic E_8 factor will correspond to a sign change for the doublet of the $SU(2)$ subgroup of $E_8 \supset E_7 \times SU(2)$. Note that the adjoint decomposes as

$$\mathbf{248} \longrightarrow (\mathbf{133}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{56}, \mathbf{2}) \in E_7 \times SU(2),$$

and that the projection acts on the $SU(2)$ representations as $\mathbf{3} \rightarrow \mathbf{3}$, $\mathbf{2} \rightarrow -\mathbf{2}$. This entails that two of our eight complex fermions are twisted by the projection and this part of the partition function reads then

$$\frac{1}{2} \sum_{\gamma, \delta=0} \frac{\vartheta[\gamma+h] \vartheta[\gamma-h] \vartheta^6[\gamma]}{\eta^8}. \quad (5.10)$$

The other E_8 factor (the bosonic realisation with Γ_8) is not affected by the projection. The full partition function for the internal theory is the product of all the above-mentioned parts:

$$C_{\text{int}} = \frac{\Gamma_{2,2}}{\eta^2 \bar{\eta}^2} \frac{\Gamma_8}{\eta^8} \frac{1}{2} \sum_{a,b} (-1)^{a+b+ab} \frac{\bar{\vartheta}[b]^a}{\bar{\eta}} \frac{1}{2} \sum_{g,h=0} Z_{(4,4)}[g] \frac{\bar{\vartheta}[b+h] \bar{\vartheta}[b-h]}{\bar{\eta}^2} \frac{1}{2} \sum_{\gamma, \delta=0} \frac{\vartheta[\gamma+h] \vartheta[\gamma-h] \vartheta^6[\gamma]}{\eta^8} \quad (5.11)$$

In (5.7) or implicitly in (5.2), $C_{\text{int}}[b]^a$ is just the above with $C_{\text{int}} =: \sum_{a,b} (-1)^{a+b+ab} C_{\text{int}}[b]^a$.

Note that the term in (5.2) with the $\bar{\tau}$ -derivative includes the non-compact fermions. We will combine these with the T^2 and $K3$ fermions and sum over their spin structures:

$$\begin{aligned} & \frac{-i}{\pi} \frac{1}{2} \sum_{\text{even}} (-1)^{a+b} \bar{\partial}_{\bar{\tau}} \left(\frac{\bar{\vartheta}[b]^a}{\bar{\eta}} \right) \frac{\bar{\vartheta}[b] \bar{\vartheta}[b+h] \bar{\vartheta}[b-h]}{\bar{\eta}^3} \\ &= \frac{1}{24 \bar{\eta}^4} \left[(\bar{\vartheta}_2^4 - \bar{\vartheta}_4^4) \bar{\vartheta}_3^2 \bar{\vartheta}[h] \bar{\vartheta}[-h] - (\bar{\vartheta}_3^4 + \bar{\vartheta}_4^4) \bar{\vartheta}_2^2 \bar{\vartheta}[g] \bar{\vartheta}[-h] + (\bar{\vartheta}_2^4 + \bar{\vartheta}_3^4) \bar{\vartheta}_4^2 \bar{\vartheta}[h] \bar{\vartheta}[-h] \right] \\ &= -\frac{1}{2} \bar{\eta}^2 \bar{\vartheta}[1-h]^2 (-1)^{1+h} \end{aligned}$$

where we have used (B.6) and (B.3); and the use of (B.4) will convince the reader that whatever the combination of g and h , the square brackets yield $-12 \bar{\eta}^6 \vartheta_{[1-g]}^{[1-h]2} (-1)^{1+h}$. In particular, this vanishes for $(g, h) = (0, 0)$.

If we multiply it with the $K3$ bosons and the uncompactified bosons ($1/\eta^2 \bar{\eta}^2$ for two transverse bosons in the LCG), we obtain, noting that $\vartheta_{[1-g]}^{[1-h]} (-1)^{1+h} = \vartheta_{[1+g]}^{[1+h]}$:

$$-8 \frac{\bar{\eta}^2}{\vartheta_{[1+g]}^{[1+h]} \vartheta_{[1-g]}^{[1-h]}}, \quad (h, g) \neq (0, 0). \quad (5.12)$$

Taking further into account the remaining E_8 fermions with orbifold projection, and summing explicitly over the g, h blocks of the orbifold:

$$\begin{aligned} -8 \frac{\bar{\eta}^2}{\eta^8} \frac{1}{4} \sum_{(h,g) \neq (0,0)} \sum_{a,b=0}^1 \frac{\vartheta_{[b]}^{[a]6} \vartheta_{[b+g]}^{[a+h]} \vartheta_{[b-g]}^{[a-h]}}{\vartheta_{[1+g]}^{[1+h]} \vartheta_{[1-g]}^{[1-h]}} &= -8 \frac{\bar{\eta}^2}{\eta^8} \frac{1}{4} \left(\frac{\vartheta_2^6 \vartheta_4^2 - \vartheta_4^6 \vartheta_2^2}{\vartheta_3^2} + \frac{\vartheta_3^6 \vartheta_2^2 + \vartheta_2^6 \vartheta_3^2}{\vartheta_4^2} - \frac{\vartheta_3^6 \vartheta_4^2 + \vartheta_4^6 \vartheta_3^2}{\vartheta_2^2} \right) \\ &= -8 \frac{\bar{\eta}^2}{\eta^8} \frac{1}{4} \frac{1}{4\eta^6} \left(\vartheta_2^4 \vartheta_4^4 (\vartheta_2^4 - \vartheta_4^4) + \vartheta_2^4 \vartheta_3^4 (\vartheta_2^4 + \vartheta_3^4) - \vartheta_3^4 \vartheta_4^4 (\vartheta_3^4 + \vartheta_4^4) \right) \\ &= \frac{\bar{\eta}^2}{\eta^{14}} E_6 \end{aligned} \quad (5.13)$$

where the last bracket yields $-2 E_6$ according to (B.10).

Finally, combining the non-projected E_8 bosons with the T^2 bosons gives us the lattice function $\Gamma_{10,2}/\eta^{10} \bar{\eta}^2$, and we obtain for the overall expression of the integrand of (5.2) without the gauge theory factor:

$$\frac{-1}{4\pi^2} \frac{1}{\eta^2 \bar{\eta}^2} \sum_{\text{even}} (-1)^{a+b} 4\pi i \bar{\partial}_{\bar{\tau}} \left(\frac{\bar{\vartheta}_{[b]}^{[a]}}{\bar{\eta}} \right) C_{\text{int}[b]}^{[a]} = \frac{\Gamma_{10,2} E_6}{\eta^{24}} = \Gamma_{2,2} \frac{E_4 E_6}{\eta^{24}}. \quad (5.14)$$

This is just the “new susy index” (5.6), obtained in a different approach (SCFT for a sigma-model) but with the gauge bundle still split into one factor with bosonic realisation and the other with fermionic one. Thus our explicit example of $T^2 \times K3$ with its two independent calculations is a verification that the integrands of (5.2) and (5.3) are indeed the same!

To give a feeling of the information content of these partition functions about the orbifold theory at hand, we show how the massless spectrum can be arrived at. We use the partition functions in (5.11) and claim that the twisted sector contains the massless states 8 **(56,1,1)** and 32 **(1,2,1)** in the $E_7 \times SU(2) \times E_8$. The remaining **(56,2,1)** + 4 **(1,1,1)** are in the untwisted sector. Now the twisted sector in (5.11) corresponds to $h = 1$ and the bosonic part of it is given by $a = 0$ (for which we have ϑ_3, ϑ_4 , *i.e.* half-integer powers of q , *i.e.* the NS sector). The right-moving fermions (including two non-compact transverse fermions $\vartheta_{[b]}^{[a]}/\eta$) give us

$$\frac{1}{2\bar{\eta}^4} (\bar{\vartheta}_3^2 \bar{\vartheta}_2^2 + \bar{\vartheta}_4^2 \bar{\vartheta}_2^2) \quad (g = 0 \text{ and } g = 1 \text{ resp.}). \quad (5.15)$$

Using this expression, we look at the lowest powers (massless states) in the full partition function

$$\begin{aligned} \frac{1}{2} \sum_{a=0,b} \frac{1}{2} \sum_{g,h=1} \dots Z_{4,4}[g] \frac{1}{2} \sum_{\gamma,\delta} \dots &= \frac{1}{4} \bar{\vartheta}_2^2 |\vartheta_2|^4 [\bar{\vartheta}_3^2 |\vartheta_3|^4 (\vartheta_2^2 \vartheta_3^6 + \vartheta_3^2 \vartheta_2^6) + \bar{\vartheta}_4^2 |\vartheta_4|^4 (\vartheta_3^2 \vartheta_2^6 - \vartheta_2^2 \vartheta_4^6)] \\ &= \frac{1}{4} 2^4 \bar{q}^{1/2} (1 + \dots) 4q^{1/4} (1 + \dots) [4q^{1/4} (16\bar{q}^{1/2} + \dots)] \\ &= 2^{10} q \bar{q}^{1/2} + \dots \end{aligned}$$

So we have 2^{10} twisted massless scalars, which is the bosonic content of 2^9 twisted massless hypermultiplets, which is just $8 \cdot \mathbf{56} + 32 \cdot \mathbf{2}$ as expected from the spectrum.

5.4 Examples of Threshold Corrections

In section 5.3, we devised a convenient way for calculating the integrand of the one-loop threshold corrections, by combining the ingredients of the partition function of the heterotic orbifold compactification. We will now proceed to incorporate the gauge theory factor, that is the trace of the Casimir operator or the square brackets in (5.2).

Let us go back to our $N=2$ \mathbb{Z}_2 orbifold $T^2 \times K3$ of section 5.3 with gauge group $E_8 \times E_7 \times SU(2)$. We shall let $\left[Q_i^2 - \frac{k_i}{4\pi\tau_2}\right]$ act on the three factors of the gauge group separately, by replacing Q_i^2 with $(\frac{\partial v_1}{2\pi i})^2|_{v_j=0}$ acting on the character $\chi_R(v_j)$ of the corresponding factor. That is, we first write the character with a v -dependence and then differentiate twice *w.r.t.* only one of the v_j , say v_1 . For instance, the E_8 character is $\chi_{E_8} = \Gamma_8/\eta^8 = E_4/\eta^8 = (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)/2\eta^8$, and with v -dependence it becomes

$$\chi_{E_8}(v_j) = \frac{1}{2} \sum_{a,b=0}^1 \frac{\prod_{j=1}^8 \vartheta_b^{[a]}(v_j)}{\eta^8}. \quad (5.16)$$

When acting with $\left[(\frac{\partial v_1}{2\pi i})^2|_{v_j=0} - \frac{k_i}{4\pi\tau_2}\right]$, the operator on the left can be replaced by $\frac{1}{i\pi}\partial_\tau$, see (B.5), with an extra factor of $1/8$ since the other theta functions are now also affected by the ∂_τ . Taking out this factor of $1/8$, we are left with the covariant derivative D_4 of (B.13), and the whole expression is, by (B.15):

$$\left[(\frac{\partial v_1}{2\pi i})^2|_{v_j=0} - \frac{k_i}{4\pi\tau_2}\right] \chi_{E_8}(v_j) = \frac{\hat{E}_2}{12} \frac{E_4 - E_6}{\eta^8}.$$

Since all other factors of the gauge group or of the internal partition function remain the same, we simply ought to replace the Γ_8/η^8 factor of (5.14) by the above result, and the complete threshold (5.2) is:

$$\Delta_{E_8} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[-\frac{1}{12} \Gamma_{2,2} \frac{\hat{E}_2 E_4 E_6 - E_6^2}{\eta^{24}} + 60 \right], \quad (5.17)$$

where we bear in mind that, as before, the lattice function $\Gamma_{2,2}$ depends on the torus moduli: $T, U \in \mathcal{H}$. We note that the (nearly) modular form in the fraction has a Fourier expansion starting with $720 + \dots$, that $\Gamma_{2,2}$ starts with $1 + \dots$, so that dividing by -12 , we do indeed obtain -60 as the constant term. This agrees with the beta function coefficient b_{E_8} of the corresponding orbifold (see appendix E.2 of [G-04]).

Similarly, for the threshold corresponding to the E_7 factor of the gauge group, we exchange the roles of the bosonic and fermionic E_8 partition functions. That is we keep Γ_8/η^8 unchanged and let $\left[(\frac{\partial v_1}{2\pi i})^2|_{v_j=0} - \frac{k_i}{4\pi\tau_2}\right]$ act on the E_7 character (5.10) with v_1 -dependence:

$$\chi_{E_7}(v_1) = \frac{1}{2} \sum_{a,b} \frac{\vartheta_b^{[a]}(v_1) \vartheta_b^{5[a]} \vartheta_{b+g}^{[a+h]} \vartheta_{b-g}^{[a-h]}}{\eta^8} \quad (5.18)$$

We remember from (5.13) that these were combined with $\bar{\eta}^2/\vartheta\vartheta$ of (5.12) and summed over g, h to yield E_6 . This is basically a product of 12 theta functions, of which only the first will have the v_1 dependence, so that we can replace the $\partial_{v_1}^2$ by $\frac{1}{12}\partial_\tau$ to have $D_6 E_6$, again according to (B.5). The covariant derivative yields $E_4^2 - \hat{E}_2 E_6$, and combining this with the remaining toroidal bosons $\Gamma_{2,2}$ and Γ_8/η^8 , we obtain the desired threshold corrections:

$$\Delta_{E_7} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left[-\frac{1}{12} \Gamma_{2,2} \frac{\hat{E}_2 E_4 E_6 - E_4^3}{\eta^{24}} - 84 \right], \quad (5.19)$$

where the fraction has a Fourier expansion starting with $-1008 + \dots$, which yields 84 when divided by 12. This again agrees with the beta function coefficient b_{E_7} of the corresponding orbifold.

Note that (5.18) contains also the character for the $SU(2)$ subgroup; so that if we were to compute the trace of the $SU(2)$ Casimir, we would obtain the same result for the threshold and consequently for $b_{SU(2)}$: 84, also agreeing with the table of beta-function coefficients in appendix E.2 of [G-04].

It is interesting to note that the difference of our two thresholds can be easily computed: recalling that $\eta^{24} = \Delta = (E_4^3 - E_6^2)/1728$, we evaluate the integral using the trick displayed in [DKL-91] (“lattice unfolding technique”):

$$\Delta_{E_8} - \Delta_{E_7} = -144 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} (\Gamma_{2,2} - 1) = 144 \log(4\pi^2 T_2 U_2 |\eta(T)\eta(U)|^4),$$

which scales as $\sim T_2$, *i.e.* as the volume of the torus, in the decompactification limit.

The above case of N=2 orbifold is closely related to the N=1 orbifold where we introduce a second \mathbb{Z}_2 twist to obtain a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with gauge group $E_8 \times E_6 \times U(1)^2$. However, the construction is independent of the untwisted moduli (T_i, U_i) of the three twisted 2-planes, and the threshold corrections are not affected by this N=1 sector. So the threshold corrections carry over from the N=2 sectors (only one 2-plane twisted): Δ_{E_8} is as in (5.17) and Δ_{E_6} as in (5.19). Consequently, the constant term of the whole integrand is 3/2 of what it used to be (3 for the three 2-planes and 1/2 due to the extra \mathbb{Z}_2 twist).

The reader will wonder what the above explicit expressions for one-loop threshold corrections to gauge couplings have to do with the Gromov-Witten invariants we started with in the first chapter. The answer was in fact already given in section 4.3: these Δ_i occur in an ODE for the prepotential F_0 , *i.e.* the genus-0 GW potential. The resolution of the integrals over the fundamental domain yield polylog expressions that can be reorganised as in (4.1). This convenient sum allowed [HM-95] to extract the coefficients $c(-\frac{r^2}{2})$, with meaning in enumerative geometry as GW invariants generally have.

A Appendix: Jacobi forms

Definition According to [EZ-85], a (holomorphic) **Jacobi form of weight k and index m** (non-negative integers) is a holomorphic function $\phi(\tau, z) : \mathcal{H} \times \mathbb{C} \mapsto \mathbb{C}$ satisfying the following three conditions:

$$\begin{aligned} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) &= (c\tau + d)^k e^{2\pi i \frac{mcz^2}{c\tau + d}} \phi(\tau, z), & \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in SL(2, \mathbb{Z}), \\ \phi(\tau, z + \lambda\tau + \mu) &= e^{-2\pi i(\lambda^2\tau + 2\lambda z)} \phi(\tau, z), & \lambda, \mu &\in \mathbb{Z}, \\ \phi(\tau, z) &= \sum_{n \geq 0} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c_{n,r} q^n y^r, & q &= e^{2\pi i\tau}, y = e^{2\pi iz}. \end{aligned}$$

The first condition is reminiscent of transformation properties of theta functions; on the divisor $z = 0$, $\phi(\tau, 0)$ is a modular form of weight k . A remarkable property is that this extends to rational portions z of the fundamental lattice $\{1, \tau\} : e^{2\pi i m \lambda^2 \tau} \phi(\tau, \lambda\tau + \mu)$ (with $\lambda, \mu \in \mathbb{Q}$) is a modular form of weight k –if not identically zero–, though in general only for a subgroup of $SL(2, \mathbb{Z})$. The accompanying behaviour of $z \mapsto \frac{z}{c\tau + d}$ follows from the postulated behaviour under the two $SL(2, \mathbb{Z})$ generators S and T : $\phi(-1/\tau, z/\tau)$ and $\phi(\tau + 1, z)$.

Periodicity The second condition similarly reproduces periodicity properties of theta functions. It entails the periodicity of the coefficients:

$$c_{n,r} = c_{n',r'} \quad \text{if } r' \equiv r \pmod{2m} \quad \text{and } n' = n + \frac{r'^2 - r^2}{4m},$$

or equivalently

$$c_{n,r} = c_{n+m-r, r-2m} = c_{n+m+r, r+2m}, \quad (\text{A.1})$$

a hallmark of Jacobi forms.

The condition $r^2 \leq 4nm$: The third condition is an interesting one; first it requires the convergence of the Fourier series, secondly it requires $r^2 \leq 4nm$. The latter requirement can be explained from three different perspectives.

Firstly, it is a consequence of the above-mentioned fact that $e^{2\pi i m \lambda^2 \tau} \phi(\tau, \lambda\tau)$ is a modular form for all $\lambda \in \mathbb{Q}$: holomorphicity at $\tau = \infty$ requires $n + r\lambda + m\lambda^2 \geq 0 \quad \forall \lambda$, i.e. $r^2 \leq 4nm$ after replacing λ by the extremum.

Secondly, $r^2 \leq 4nm$ is the condition for the matrix $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ to be positive semi-definite, $m, n, r \in \mathbb{Z}$. If we sum over such semi-integral 2×2 matrices in the Fourier-expansion

$$F(Z) := \sum_{T \geq 0} c_T e^{2\pi i \operatorname{tr} TZ},$$

we obtain a so-called *Siegel modular form* of degree 2, that is a holomorphic function of the generalised upper-half plane \mathcal{H}_2 of all symmetric 2×2 matrices with positive definite imaginary part, i.e. $\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathcal{H}, z \in \mathbb{C}$, covariant under the action of the Siegel modular group $Sp(4, \mathbb{Z})$:

$$F\left(\frac{AZ + B}{CZ + D}\right) = \det(CA + D)^k F(Z), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z})$$

for some weight k . Note that $\operatorname{tr} TZ = n\tau + rz + m\tau'$ and that we may as well rewrite the Fourier expansion as

$$F(\tau, z, \tau') = \sum_{m \geq 0} \left(\sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4nm}} c_T q^n y^r \right) e^{2\pi i m \tau'},$$

in which the bracketed piece is a Jacobi form of weight k and index m . In this larger context of Siegel modular forms, we see the origin of the condition $r^2 \leq 4nm$.

A third way to understand the condition is through an example of a Jacobi form: a theta function, defined by summing over the roots x of an even, self-dual lattice Γ of rank N :

$$\phi(\tau, z) := \sum_{x \in \Gamma} q^{x^2/2} y_0^x, \quad y_0^x = e^{2\pi i(x_0 \cdot x)z}, x_0 \in \Gamma.$$

This is a form of weight $k = N/2$ and index $m = x_0^2/2$, and the Schwarz inequality for the bilinear form associated to Γ reads $(x \cdot x_0)^2 \leq x^2 x_0^2$, that is $r^2 \leq 4nm$. For more general lattices (not necessarily self-dual), the form will be covariant under only a congruence subgroup of $SL(2, \mathbb{Z})$.

Zeros For fixed τ , a simple contour integration shows that the the number of zeros minus the number of poles in a fundamental domain for $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ is given by $2m$. Since our Jacobi forms are assumed holomorphic, this is just the number of zeros. If m vanishes, the second condition requires ϕ to be doubly-periodic –hence constant– in z , that is, ϕ is a mere modular form. Had we allowed m to be negative, we would have *meromorphic* Jacobi forms – examples of which are inverses of Jacobi forms.

Decomposition The periodicity of the coefficients entails that we can write a Jacobi form as a finite linear combination of theta functions with modular forms as coefficients. Indeed, (A.1) tells us that the $c_{n,r}$ only depend on $4mn - r^2$ and on $r \pmod{2m}$. Thus, for a fixed residue class $\mu \in \mathbb{Z}/2m\mathbb{Z}$, all $c_{n,r}$ with $r \equiv \mu \pmod{2m}$ only depend on $N := 4mn - r^2$; for these we define $c_\mu(N) := c_{4nm-r^2, r}$. We then have $2m$ functions

$$\begin{aligned} h_\mu(\tau) &:= \sum_{N \geq 0} c_\mu(N) q^{N/4m}, \quad \mu \in \mathbb{Z}/2m\mathbb{Z} \\ &= q^{-\mu^2/4m} \sum_{n \geq 0} c_{n,\mu} q^n, \end{aligned}$$

wherein it is understood that $c_\mu(N) = 0$ if $N \not\equiv -\mu^2 \pmod{4m}$. These are vector-valued modular forms of weight $k - 1/2$ (with $2m$ components), which, under modular transformations, transform into linear combinations of themselves. They allow us to rewrite the Jacobi form ϕ in a basis of theta functions:

$$\begin{aligned} \phi(\tau, z) &= \sum_{\mu \pmod{2m}} h_\mu(\tau) \theta_{m,\mu}(\tau, z) \\ \theta_{m,\mu}(\tau, z) &:= \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu \pmod{2m}}} q^{r^2/4m} y^r \end{aligned} \tag{A.2}$$

The theta functions similarly transform under the Jacobi group into linear combinations of themselves, with weight $1/2$ and index m (so they are not Jacobi forms strictly speaking). This decomposition of Jacobi form holds also for forms of half-integer weight (shall not deal with them), as well as for weak Jacobi forms (see below). In the latter case h_μ may contain negative (fractional) powers of q , and $N \geq 0$ ought to read $N \geq -m^2$; the $c_\mu(N)$ vanish anyway when $N < -r^2$ (where $|r| \leq m$, $r \equiv \mu \pmod{2m}$).

Weak Jacobi forms If the requirement $r^2 \leq 4nm$ is dropped, one can still prove that the coefficients $c_{n,r}$ vanish unless $r^2 \leq 4nm + m^2$, in which case we have a so-called *weak* Jacobi form (still holomorphic). The periodicity of the coefficients still holds, as does the property of decomposition into a linear combination of theta function.

In the weak case, the weight k can be negative but no less than $-2m$ (m still assumed positive). If this holds, $\phi(\tau, 0)$ must vanish identically, as there are no holomorphic modular forms of negative weight (safe perhaps for proper subgroups of $SL(2, \mathbb{Z})$). The same is true for Jacobi forms of odd weight, whether weak or not. In fact, the space of weak Jacobi forms of weight k and index m is isomorphic to a direct sum of vector spaces of modular forms of different weights: $M_k \oplus \dots \oplus M_{k+2m}$ for k even, while for k odd the subscripts run from $k+1$ to $k+2m-3$. (Recall that $M_k = \emptyset$ for k odd.) Better still, for k even the correspondence permits us to write a weak Jacobi form as

$$\phi_{k,m} = \sum_{i=0}^m f_i A_{-2,1}^i B_{0,1}^{m-i}$$

for two generators $A_{-2,1}, B_{0,1}$ of the ring of weak Jacobi forms of even weight (which is thus a polynomial algebra over M_*), and $f_i \in M_{k+2i}$. Note that since M_0 contains the constant functions while M_2 is empty, we see that $\phi_{0,1}$ is unique (up to a constant) and equals $B_{0,1}$. Furthermore,

$$A_{-2,1} = -\frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} \quad \text{and} \quad B_{0,1} = -\frac{3}{\pi^2} \wp A_{-2,1}$$

where ϑ_1 is the first Jacobi theta function, with a simple zero at $z = 0$ (ϑ_1^2 transforms as a Jacobi form of weight 1, index 1, but with multiplier $-i$), and \wp is the Weierstrass function, itself a meromorphic Jacobi form of weight -2 and index 0 (with a double pole at $z = 0$). A and B are both constant functions at $z = 0$, equal to 0 and 12 respectively.

See appendix A of [G-04] for a fuller treatment.

B Appendix: Theta Functions and Modular Forms

This is drawn in part from the exhaustive Appendix A and F of [K-97].

Theta functions

$$\begin{aligned} \vartheta_1(v|\tau) &= -i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+\frac{1}{2})^2/2} y^{n+\frac{1}{2}} &= 2q^{\frac{1}{8}} \sin[\pi v] \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n y)(1 - q^n y^{-1}) \\ \vartheta_2(v|\tau) &= \sum q^{(n+\frac{1}{2})^2/2} y^{n+\frac{1}{2}} &= 2q^{\frac{1}{8}} \cos[\pi v] \prod (1 - q^n)(1 + q^n y)(1 + q^n y^{-1}) \\ \vartheta_3(v|\tau) &= \sum q^{n^2/2} y^n &= \prod (1 - q^n)(1 + q^{n-1/2} y)(1 + q^{n-1/2} y^{-1}) \\ \vartheta_4(v|\tau) &= \sum (-1)^n q^{n^2/2} y^n &= \prod (1 - q^n)(1 - q^{n-1/2} y)(1 - q^{n-1/2} y^{-1}) \end{aligned} \tag{B.1}$$

v -periodicity formulae

$$\begin{aligned} \vartheta_b^{[a]}(v + \tfrac{1}{2}) &= \vartheta_{b-1}^{[a]}(v) \\ \vartheta_b^{[a]}(v + \tfrac{\tau}{2}) &= i^b q^{-1/8} y^{-1/2} \vartheta_b^{[a-1]}(v) \\ \vartheta_b^{[a]}(v + \tfrac{1+\tau}{2}) &= -i^{b+1} q^{-1/8} y^{-1/2} \vartheta_{b-1}^{[a-1]}(v) \\ \vartheta_b^{[a]}(v + 1) &= (-1)^a \vartheta_b^{[a]}(v) \\ \vartheta_b^{[a]}(v + \tau) &= (-1)^b q^{-1/2} y^{-1} \vartheta_b^{[a]}(v) \end{aligned}$$

Useful identities

$$\vartheta_2 = 2 \frac{\eta(2\tau)}{\eta} \quad \vartheta_3 = \frac{\eta^5}{\eta(2\tau)^2 \eta(\frac{\tau}{2})^2} \quad \vartheta_4 = \frac{\eta(\frac{\tau}{2})^2}{\eta} \tag{B.2}$$

$$\begin{aligned}
\vartheta_2 \vartheta_3 \vartheta_4 &= 2 \eta^3 \\
\vartheta_3(z|\tau) \vartheta_3(z'|\tau) + \vartheta_2(z|\tau) \vartheta_2(z'|\tau) &= \vartheta_3\left(\frac{z+z'}{2}\middle|\frac{\tau}{2}\right) \vartheta_3\left(\frac{z-z'}{2}\middle|\frac{\tau}{2}\right) \\
\vartheta_3(z|\tau) \vartheta_3(z'|\tau) - \vartheta_2(z|\tau) \vartheta_2(z'|\tau) &= \vartheta_4\left(\frac{z+z'}{2}\middle|\frac{\tau}{2}\right) \vartheta_4\left(\frac{z-z'}{2}\middle|\frac{\tau}{2}\right) \\
\vartheta_2(v|\tau)^4 - \vartheta_1(v|\tau)^4 &= \vartheta_3(v|\tau)^4 - \vartheta_4(v|\tau)^4
\end{aligned} \tag{B.3}$$

For $v = 0$, the latter is but Jacobi's *abstruse identity*:

$$\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4. \tag{B.4}$$

Heat equation

$$\left[\frac{1}{(2\pi i)^2} \frac{\partial^2}{\partial v^2} - \frac{1}{i\pi} \frac{\partial}{\partial \tau} \right] \vartheta_{[b]}^{[a]}(v|\tau) = 0, \tag{B.5}$$

as well as

$$\partial_\tau \log \frac{\vartheta_2}{\eta} = \frac{i\pi}{12} (\vartheta_3^4 + \vartheta_4^4), \tag{B.6}$$

and more generally for $(a, b) \neq (1, 1)$:

$$\partial_\tau \log \frac{\vartheta_{[b]}^{[a]}}{\eta} = \frac{i\pi}{12} \left(\vartheta_{[b]}^{[a+1]} - \vartheta_{[b+1]}^{[a]} + (-1)^b \vartheta_{[b+1]}^{[a+1]} \right). \tag{B.7}$$

Modular Forms

We list the first few Eisenstein series:

$$E_2 = \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} = 1 - 24 \sum_{n=1}^{\infty} \frac{q^n}{(1 - q^n)^2}, \tag{B.8}$$

$$E_4 = \frac{1}{2} (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \tag{B.9}$$

$$E_6 = \frac{1}{2} (\vartheta_2^4 + \vartheta_3^4) (\vartheta_3^4 + \vartheta_4^4) (\vartheta_4^4 - \vartheta_2^4) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \tag{B.10}$$

The E_4 and E_6 modular forms have weight four and six, respectively, and generate the ring of modular forms. However, E_2 is not exactly a modular form, but

$$\hat{E}_2 = E_2 - \frac{3}{\pi \tau_2} \tag{B.11}$$

is a modular form of weight 2 (though not holomorphic anymore). The (modular-invariant) j function and η^{24} can be written as

$$j = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + \dots, \quad \eta^{24} = \frac{1}{2^6 \cdot 3^3} [E_4^3 - E_6^2]. \tag{B.12}$$

We will also introduce the covariant derivative on modular forms:

$$F_{d+2} = \left(\frac{i}{\pi} \partial_\tau + \frac{d/2}{\pi \tau_2} \right) F_d \equiv D_d F_d. \tag{B.13}$$

F_{d+2} is a modular form of weight $d+2$ if F_d has weight d . The covariant derivative introduced above has the following distributive property:

$$D_{d_1+d_2} (F_{d_1} F_{d_2}) = F_{d_2} (D_{d_1} F_{d_1}) + F_{d_1} (D_{d_2} F_{d_2}). \tag{B.14}$$

The following relations and (B.14) allow the computation of any covariant derivative

$$D_2 \hat{E}_2 = \frac{1}{6} E_4 - \frac{1}{6} \hat{E}_2^2, \quad D_4 E_4 = \frac{2}{3} E_6 - \frac{2}{3} \hat{E}_2 E_4, \quad D_6 E_6 = E_4^2 - \hat{E}_2 E_6. \tag{B.15}$$

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